DISSERTATION

TOWARD A TYPE B_N GEOMETRIC LITTLEWOOD-RICHARDSON RULE

Submitted by

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In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2007

COLORADO STATE UNIVERSITY

June 7, 2007

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY DIANE E. DAVIS ENTITLED "TOWARD A TYPE B_N GEOMETRIC LITTLEWOOD-RICHARDSON RULE" BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

TOWARD A TYPE B_N GEOMETRIC LITTLEWOOD-RICHARDSON RULE

We conjecture a geometric Littlewood-Richardson Rule for the maximal orthogonal Grassmannian and make significant advances in the proof of this conjecture. We consider Schubert calculus in the presence of a nondegenerate symmetric bilinear form on an odd-dimensional vector space (the type B_n setting) and use degenerations to understand intersections of Schubert varieties in the odd orthogonal Grassmannian. We describe the degenerations using combinatorial objects called checker games. This work is closely related to Vakil's Geometric Littlewood-Richardson Rule (Annals of Mathematics, **164**).

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ACKNOWLEDGEMENTS

I would like to take this opportunity to acknowledge several people, without whom, completion of this dissertation would not have occurred.

My parents, Paul and Judy Davis, and my siblings, Ann, Marty, John, Elaine, Bob, and Jane. You are the most important. Thank you for your unconditional love and support.

There are four committee member signatures on this dissertation. I would like however, to thank all the faculty who have given their time to be a part of my committee: Jeanne Duflot, Michael Kirby, Holger Kley, Ross McConnell, Rick Miranda, and Louis Scharf.

Ravi Vakil, thank you for support and motivation for this work.

Rick Miranda, thank you for encouraging me to pursue a PhD and for your advice throughout my years at CSU.

My passion for mathematics is due in large part to these inspirational teachers from Muhlenberg High School, Bloomsburg University of Pennsylvania, and Colorado State University:
Valetta Eshbach, Carl Richie, Scott Inch, Darel Hardy, and Jeanne Duflot. Thank you.

Finally, I'd like to give special thanks to my advisor, Holger Kley. Thank you Holger for being such a supportive advisor and mentor. Thank you for taking me on as a student, for teaching me so much, and for helping me see this project to the end. You are an awesome teacher and role model.

Additionally, the National Science Foundation grant

NSF DMS/DCCF 0434351: New Tools for Algebro-Geometric Data Analysis

supported one year of research for this project.

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Chapter 1

BACKGROUND

1.1 Littlewood-Richardson Numbers

Littlewood-Richardson numbers occur in a variety of contexts. In combinatorics, the Littlewood-Richardson number $c^{\nu}_{\lambda\mu}$ counts, for example, the number of skew tableaux of shape ν/λ whose rectification is of shape μ [6]. In algebra, the ring of symmetric functions has a basis of Schur functions. The Schur functions can be enumerated using partitions and the structure coefficients of the ring are the Littlewood-Richardson numbers. In representation theory, the general linear group $GL_n(\mathbb{C})$ has irreducible representations enumerated by partitions. The tensor product of two irreducible representations can be decomposed into a sum of irreducible representations with multiplicities that are Littlewood-Richardson numbers [21]. Finally, the Littlewood-Richardson numbers play a role in geometry, the context in which this dissertation is set.

1.2 Geometry Discussion

The usual, or type A_n, Grassmann manifold $G(k,n)$ is the space of k-dimensional vector subspaces of $\mathbb{C}^n$.

Example 1.2.1. $G(1,3)$ is the set of all lines through the origin in $\mathbb{C}^3$ (this is the projective plane, $\mathbb{P}^2$). Consider a set of reference spaces, called a flag, in $\mathbb{C}^3$, consisting of the origin, a line through the origin, a plane containing that line, and all of three-space, denoted

$$F_. = (F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq F_3 = \mathbb{C}^3).$$

We can ask what lines through the origin meet these reference spaces in a certain way. For example, we might ask what lines meet the reference plane F_2 in one dimension? The solution to this question is a subspace of the Grassmannian $G(1,3)$, the set of lines through the origin that lie in the reference plane F_2.

In general, the space of all k-dimensional subspaces of $\mathbb{C}^n$ which intersect certain reference spaces in a predetermined way is a locally closed subvariety of the Grassmannian.

Definition 1. Given $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = k$ and a flag $F_. = (F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n)$ of subspaces, the set of all $V \in G(k,n)$ such that $\dim V \cap F_j = \alpha_j$ for $1 \leq j \leq n$, is called a *Schubert cell.*

In example 1.2.1, $(\alpha_1, \alpha_2, \alpha_3) = (0,1,1)$.

A Schubert condition on $\mathbb{C}^n$ is encoded by a partition λ with at most $n-k$ columns and k rows. We denote the corresponding Schubert cell $\Omega_\lambda(F_.)$. The vector space V is an element of the Schubert cell $\Omega_\lambda(F_.)$ if $\dim(V \cap F_{(n-k)+j-\lambda_j}) = j$ for $1 \leq j \leq k$.

Listing $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = k$ for a Schubert condition is equivalent to giving a partition λ via the following bijection:

- Given $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$, we have $\dim(V \cap F_{(n-k)+j-\lambda_j}) = j$ for $1 \leq j \leq k$.

- Given $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n = k$ then for each j, $1 \leq j \leq n$, where $\alpha_{j-1} < \alpha_j$ (with the convention that $\alpha_0 = 0$) we have $\lambda_{\alpha_j} = (n-k) + \alpha_j - j$.

In example 1.2.1, the required intersection is described by the partition $\lambda = (1) = \square$.

	F_0	F_1	F_2	F_3	F_4	F_5
$\dim(V \cap F_j)$	0	0	1	1	2	2

Table 1.1: Dimensions of intersection for Example 1.2.2

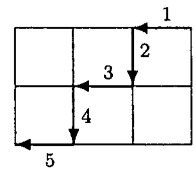


Figure 1.1: Walk corresponding to dimensions in Table 1.1

Example 1.2.2. Consider the Grassmannian $G(2,5)$ and a fixed flag $F.$ with restrictions on $\dim(V \cap F_j)$ given by Table 1.1. The corresponding partition λ is then found by the two calculations:

1. $\dim(V \cap F_{(5-2)+1-\lambda_1}) = 1$ where
$$F_{(5-2)+1-\lambda_1} = F_2 \Rightarrow \lambda_1 = 2$$

2. $\dim(V \cap F_{(5-2)+2-\lambda_2}) = 2$ where
$$F_{(5-2)+2-\lambda_2} = F_4 \Rightarrow \lambda_2 = 1$$

So $\lambda = {\tiny\yng(2,1)}$ which gives the Schubert cell $\Omega_{\tiny\yng(2,1)}(F.)$.

A third equivalent way to describe a Schubert condition is with an n-step walk through the vertices of a $k \times (n-k)$ grid of boxes. Beginning from the northeast corner, walk either west or south. On the j^{th} step, move west if $\dim(V \cap F_j) = \dim(V \cap V_{j-1})$ and move south if $\dim(V \cap F_j) = \dim(V \cap F_{j-1}) + 1$. The partition λ is then the collection of boxes northwest of the n-step walk.

In Table 1.1, jumps occur at F_2 and F_4. The five step walk is given in Figure 1.1.

The closure $\overline{\Omega_\lambda}(F.)$ of a Schubert cell is called a *Schubert variety.* This is the set of V that meet $F.$ at least in the prescribed way, or more precisely, $\dim(V \cap F_j) \geq \alpha_j$ for all j. A natural question is:

Question 1.2.1. *How do two Schubert varieties intersect?*

An important special case is:

Question 1.2.2. *If a collection of Schubert varieties has a finite number of points in common, what exactly is this number?*

Such questions are typical Schubert calculus questions. We will concern ourselves with answers to generalizations of these questions. More precisely, $\operatorname{codim}_{G(k,n)}(\overline{\Omega_\lambda}(F_.)) = |\lambda|$, so if we have r flags $F^1_., F^2_.,\ldots, F^r_.$, and the flags are in general position, then

$$\operatorname{codim}_{G(k,n)}(\bigcap_{j=1}^{r} \overline{\Omega}_{\lambda^j}(F^j_.)) = \sum_{j=1}^{r} |\lambda^j|$$

where λ^j is a partition for $1 \leq j \leq r$. A general *Schubert problem* consists of finding all V which satisfy the Schubert conditions λ^j with respect to $F^j_.$ for $j = 1, \ldots, r$.

By the Kleiman-Bertini theorem, we can categorize the number of solutions possible:

1. If $\sum_{j=1}^{r} |\lambda^j| < k(n-k)$, then there are infinitely many solutions (over $\mathbb{C}$).

2. If $\sum_{j=1}^{r} |\lambda^j| > k(n-k)$, and $F^1, F^2,\ldots, F^r$ are in general position, then there are no solutions, i.e. $\bigcap_{i=1}^{r} \overline{\Omega}_{\lambda^j}(F^j_.) = \emptyset$.

3. If $\sum_{j=1}^{r} |\lambda^j| = k(n-k)$, and $F^1, F^2,\ldots, F^r$ are in general position, then there are a fixed, finite number of solutions, i.e. the set $\{V \in \bigcap_{i=1}^{r} \overline{\Omega}_{\lambda^j}(F^j_.)\}$ is finite.

1.2.1 Orthogonal (type B_n) flags

The setting described above for the usual Grassmannian is called type A_n. If we place a non-degenerate symmetric bilinear form B on the odd-dimensional vector space $\mathbb{C}^{2n+1}$ and work with isotropic subspaces (see definition 2), we are describing a type B_n setting. *Caution*: The form B is *not* an inner product, so there exist nonzero vectors v with $B(v, v) = 0$.

The following is based on [7]. Since we are working over an algebraically closed field, we can always choose a basis $e_1, e_2, \ldots e_{2n+1}$ for $\mathbb{C}^{2n+1}$ such that

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i + j = 2n+2 \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$B(x, y) = \begin{bmatrix} x_1 & \ldots & x_{2n+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & \ldots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{2n+1} \end{bmatrix}$$

for $x, y \in \mathbb{C}^{2n+1}$.

For a subspace $V \subset \mathbb{C}^{2n+1}$, define $V^{\perp}$ as

$$V^{\perp} = \{x \in \mathbb{C}^{2n+1} \quad | \quad B(x,y) = 0 \quad \forall y \in V\}.$$

The perp space $V^{\perp}$ has dimension $\dim V^{\perp} = 2n + 1 - \dim V$.

Definition 2. A subspace $V \subset \mathbb{C}^{2n+1}$ is called *isotropic* if $B(x, y) = 0$ for all $x, y \in V$. A *maximal* isotropic subspace of $\mathbb{C}^{2n+1}$ is an isotropic subspace whose dimension is n.

The following facts are immediate consequences from the properties of the form B:

1. $(V^{\perp})^{\perp} = V$

2. V is isotropic if and only if $V \subset V^{\perp}$.

3. We always consider subspaces of $\mathbb{C}^{2n+1}$ with the induced (restricted) form of B. If $V \subset \mathbb{C}^{2n+1}$ then $B|_V$ is symmetric bilinear (but not necessarily nondegenerate).

4. V is isotropic if and only if $B|_V = 0$

5. V is isotropic or the perp of an isotropic space if and only if $\mathrm{rank}(B|_V)$ is minimal (minimal means $\max\{0, 2\dim V - (2n + 1)\}$).

6. If V is the perp of an isotropic space, then $\mathrm{rank}(B|_V)$ is odd.

7. If $V \subset W \subset \mathbb{C}^{2n+1}$ then B induces a well-defined form on W/V if and only if V is isotropic and $W \subset V^{\perp}$. In particular, if V is isotropic, then B induces a non-degenerate form on $V^{\perp}/V$.

With these facts, we can construct isotropic flags $F_{\cdot} = (F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n \subsetneq \mathbb{C}^{2n+1})$ with $\dim F_i = i$ and F_i isotropic. Specifically, choose F_1 to be a one-dimensional isotropic subspace of $\mathbb{C}^{2n+1}$. Once we choose F_1, we choose an isotropic line $F_2/F_1 \subset F_1^{\perp}/F_1$ and continue in the same manner, with $F_i/F_{i-1} \subset F_{i-1}^{\perp}/F_{i-1}$ for $i = 1, \ldots, n$. We can complete this flag by setting $F_{n+i} = F_{n+1-i}^{\perp}$ for $1 \leq i \leq n+1$. An alternate way of building $F_{\cdot}$ is to choose a maximal isotropic subspace, $F_n \subset \mathbb{C}^{2n+1}$. Then choose a flag $F_1 \subsetneq \cdots \subsetneq F_n$. In this method, once a maximal isotropic space F_n is chosen, all subspaces of F_n are immediately isotropic. The flag is completed by setting $F_{n+i} = F_{n+1-i}^{\perp}$ for $1 \leq i \leq n+1$. The variety of all such flags is $OFl(2n+1)$.

Matrix representation of a flag

A flag $E. \in OFl(2n+1)$ can be described by a matrix representation where each row r_i is a vector and $E_i = \langle r_1, \ldots, r_i \rangle$ for $1 \leq i \leq n$. We need not include rows greater than n since these spaces are determined by $E_1, \ldots, E_n$.

Attitude table for a flag cell

$OFl(2n+1)$ has a cell structure. A cell can be represented using an attitude table. Fix a flag $F. = (F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n)$ and choose a basis such that $F_i = \langle e_1, \ldots, e_i \rangle$. We want to describe the cell of $OFl(2n+1)$ that contains all flags $E.$ such that $\dim(E_i \cap F_j) = \alpha_{i,j}$ for given $\alpha_{i,j}$ with $1 \leq i \leq n$, and $1 \leq j \leq 2n+1$. For the cell to be nonempty, it is necessary that

1. $\alpha_{i+1,j} - \alpha_{i,j} \leq 1$
2. $\alpha_{i,j+1} - \alpha_{i,j} \leq 1$
3. $\alpha_{n,2n+1} = n$ and
4. if $i \leq k$ and $j \leq l$ then $\alpha_{i,j} \leq \alpha_{k,l}$.

The $\alpha_{i,j}$ can be written in an attitude table. An example of such a table for $n = 2$ is given in Table 1.2.

Let the n-tuple $(j_1, j_2, \ldots, j_n)$ describe the *jumping numbers* for the cell, where j_i is the first position in row i (left to right) where there is an increase from the corresponding column entry in row $i-1$. The jumping number n-tuple for the cell described in Table 1.2 is $(j_1, j_2) = (2, 5)$.

	F_0	F_1	F_2	F_3	F_4	F_5
E_0	0	0	0	0	0	0
E_1	0	0	1	1	1	1
E_2	0	0	1	1	1	2

Table 1.2: A flag attitude table for $n = 2$. Typically, the E_0 row and F_0 column are not written.

Matrix representation of a flag cell

An $n \times (2n+1)$ matrix representation can also be used to describe a cell of $OFl(2n+1)$. For each jumping number j_i, let entry (i, j_i) of the matrix be 1. Each entry in the column below a 1 or in the row to the right of a 1 will be designated as 0. The remaining entries are $*$'s, representing free entries. Isotropy conditions must be imposed, which determines some entries. After satisfying isotropy conditions, the number of free entries in the matrix is the dimension of the cell in $OFl(2n+1)$.

Example 1.2.3. The set of $E.$ that satisfy the attitude table 1.2 are given by the matrix representation

$$\begin{bmatrix} * & 1 & 0 & 0 & 0 \\ \bullet & 0 & * & \bullet & 1 \end{bmatrix}$$

where $*$ represents a free entry and $\bullet$ represents a determined entry. Determined entries come from isotropy conditions. In this example, before isotropy conditions are applied, $E.$ is represented by the matrix

$$\begin{bmatrix} a & 1 & 0 & 0 & 0 \\ b & 0 & c & d & 1 \end{bmatrix}$$

where a, b, c, d are free entries. Let row 1 be v_1 and row 2 be v_2. In order for $E.$ to be isotropic, the symmetric bilinear form imposes conditions on the variables.

$$\begin{aligned} B(v_1, v_1) &= 0 \iff & 0 &= 0 \\ B(v_1, v_2) &= 0 \iff & a + d &= 0 \\ B(v_2, v_2) &= 0 \iff & 2b + c^2 &= 0. \end{aligned}$$

For the isotropy condition imposed by rows i and j, $i \leq j$, there will be at most two linear monomials. We solve for the variable in the linear term that comes from row j. In this example, we solve for d and b. Now $d = -a$ and $b = -\frac{c^2}{2}$ are determined, so this cell has dimension $= 2$.

Permutation representation of a flag cell

A third way to represent a cell of $OFl(2n+1)$ is by a permutation. Schubert cells of $OFl(2n+1)$ correspond to elements in the Weyl group

$$W = \{\omega \in S_{2n+1} \mid \omega(i) + \omega(2n+2-i) = 2n+2 \quad \forall i\} \tag{1.1}$$

where S_{2n+1} is the symmetric group on $2n+1$ elements. Given a permutation $\omega \in W$, we produce an $n \times (2n+1)$ matrix representation of the corresponding cell by placing 1's in positions $(i, \omega(i))$ for $1 \leq i \leq n$, zeroes to the right and below the 1's, and $*$'s elsewhere. Checking the isotropy conditions (like Example 1.2.3) determines where $*$'s are replaced with $\bullet$'s. Conversely, given a $n \times (2n+1)$ matrix representation of a cell, ω is determined by looking at the position of the 1's in the matrix. The column position of the 1 in row i is $\omega(i)$ for $1 \leq i \leq n$. The rest of ω is then determined: $\omega(n+1) = n+1$ and $\omega(j) = 2n+2-\omega(2n+2-j)$ for $n+1 < j \leq 2n+1$.

1.2.2 Odd orthogonal (type B_n) Grassmannians

We now define the *(maximal) odd orthogonal Grassmannian,*

$$OGr(n, 2n+1) = \{V \in G(n, 2n+1) \mid V \text{ is isotropic wrt } B\}.$$

$OGr(n, 2n+1)$ is a homogeneous space of the orthogonal group. Like the usual (type A_n) Grassmannian, the orthogonal Grassmannian has a cell structure.

Attitude table for a cell of $OGr(n, 2n+1)$

For a fixed choice of $F_. \in OFl(2n+1)$, a cell of $OGr(n, 2n+1)$ can be described by a single row attitude table with entries α_i, $1 \leq i \leq 2n+1$, where $\dim(V \cap F_i) = \alpha_i$. Similar to the flag attitudes, it is a necessary condition for nonempty cells that $\alpha_i \leq \alpha_j$ for $i < j$ and that $\alpha_{i+1} - \alpha_i \leq 1$. An example of such an attitude for $n = 2$ can be found in Table 1.3. Note that an

	F_1	F_2	F_3	F_4	F_5
V	0	1	1	1	2

Table 1.3: an orthogonal Grassmannian attitude table for $n = 2$

attitude table for $OFl(2n+1)$ gives more information than an attitude table for $OGr(n, 2n+1)$. In particular, there is a forgetful map

$$OFl(2n+1) \to OGr(n, 2n+1)$$

which maps $E_. \mapsto E_n$. The attitude table for a cell of the orthogonal Grassmannian is the same as the n^{th} row of attitude tables of more than one cell in $OFl(2n+1)$. Two different cells of $OFl(2n+1)$ that have the same n^{th} row in the attitude table have the same image in $OGr(n, 2n+1)$ under the forgetful map.

Example 1.2.4. Let $F_. \in OFl(2n+1)$ be the standard flag defined by $F_j = \langle e_1, \ldots, e_j \rangle$ for $1 \leq j \leq 2n+1$. The attitudes for cells $A, B \subset OFl(5)$ are given in Tables 1.4 and 1.5 respectively. The cells are not the same, $A \neq B$, but for $E_. \in A$ there is an $E'_. \in B$ such that $E_2 = E'_2$ and vice versa. Here, $E'_2 = \langle ae_1 - \frac{c^2}{2}e_2 + ce_3 + e_4, e_1 \rangle$ for some choice of $a, c \in \mathbb{C}$ and $E_2 = \langle e_1, \frac{-g^2}{2}e_2 + ge_3 + e_4 \rangle$ for some choice of $g \in \mathbb{C}$. So for $c = g$ we have $E_2 = E'_2$. On the other hand, $E_1 = \langle e_1 \rangle$ which cannot be $E'_1 = \langle ae_1 - \frac{c^2}{2}e_2 + ce_3 + e_4 \rangle$ for any choice of a and c.

	F_1	F_2	F_3	F_4	F_5
E_1	1	1	1	1	1
E_2	1	1	1	2	2

Table 1.4: a flag attitude table for cell A

	F_1	F_2	F_3	F_4	F_5
E_1'	0	0	0	1	1
E_2'	1	1	1	2	2

Table 1.5: a flag attitude table for cell B

Matrix representation for a cell of $OGr(n, 2n+1)$

Matrix representations of two different cells of $OFl(2n+1)$ both having 1's in columns $\{j_1, j_2, \ldots, j_n\}$ will describe a unique cell of $OGr(n, 2n+1)$ with jumping numbers $\{j_1, \ldots, j_n\}$. In fact, two matrix representations of cells in the flag variety will row reduce to the same form if they describe the same cell in $OGr(n, 2n+1)$.

Lemma 1.2.1. *Elementary row operations preserve isotropy.*

Proof. For an isotropic subspace V, let $v_i, v_j \in V$ where v_i and v_j are the ith and jth rows of an $n \times (2n+1)$ matrix spanning V. $B(v_i, v_j) = 0$. Adding a multiple of row j to row i, we have the new row $v_i + \alpha v_j$.

$$\begin{aligned} B(v_i + \alpha v_j, v_i + \alpha v_j) &= B(v_i, v_i) + B(v_i, \alpha v_j) + B(\alpha v_j, v_i) + B(\alpha v_j, \alpha v_j) \\ &= B(v_i, v_i) + \alpha B(v_i, v_j) + \alpha B(v_j, v_i) + \alpha^2 B(v_j, v_j) \\ &= 0 + \alpha 0 + \alpha 0 + \alpha^2 0 \\ &= 0. \end{aligned}$$

So $v_i + \alpha v_j$ is orthogonal to itself. Similarly we can check that $B(v_i + \alpha v_j, v_k) = 0$ for all $v_k \in V$. □

Definition 3. The *reduced row echelon form* for a matrix representation of a cell of $OFl(2n+1)$ is an $n \times (2n+1)$ matrix where the 1's are arranged in descending order (left to right) and there are zeroes below each 1.

Example 1.2.5.

$$\begin{bmatrix} \bullet & * & * & * & 1 \\ \bullet & 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row reduces to}} \begin{bmatrix} \bullet & 1 & 0 & 0 & 0 \\ \bullet & 0 & * & * & 1 \end{bmatrix}$$

The set of matrix representations of cells of $OFl(2n+1)$ that are in reduced row eschelon form are in one-to-one correspondence with the cells of $OGr(n, 2n+1)$.

Permutation representation for a cell of $OGr(n, 2n+1)$

For a fixed $F_{.}$, there is a correspondence between the set of cells of $OGr(n, 2n+1)$ and a subset W' of the Weyl group of $OFl(2n+1)$:

$$W' = \{\omega \in W \mid \omega(1) < \omega(2) < \cdots < \omega(n) \text{ and } \omega(i) \notin \{2n+2-\omega(j) \text{ for } j < i\}\} \tag{1.2}$$

Lemma 1.2.2. *There are 2^n cells in $OGr(n, 2n+1)$.*

Proof. We choose the numbers to be placed in the first n positions of the Weyl group element (in table form). The i^{th} position has $2n-2(i-1)$ possibilities (there are $2n$ choices for the first slot, then $2n-2$ choices for the second slot since we cannot use $\omega(1)$ or its pair $2n+2-\omega(1)$). Thus there are

$$2n(2n-2)(2n-4)\ldots(2n-(2n-2)) = 2^n n!$$

choices for the first n positions. Since we only want the elements where $\omega(1) < \omega(2) < \cdots < \omega(n)$, we divide by the number of orderings of each collection. So the number of cells in $OGr(n, 2n+1)$ is $\frac{2^n n!}{n!} = 2^n$. □

An alternate notation for elements of W' described by F. Sottile is often useful. Shift the numbering of permutations so that $n+1$ is now 0. Then we permute $-n, \ldots, 0, \ldots, n$ instead of $1, \ldots, 2n+1$. We use the overbar to denote a negative number; for example, $-3 = \overline{3}$. In this notation,

$$W = \{\omega \in S_{\{-n,\ldots,n\}} \mid \omega(i) + \omega(\overline{i}) = 0 \quad \forall i\}$$

and

$$W' = \{\omega \in W \mid \omega(\overline{n}) < \omega(\overline{n-1}) < \cdots < \omega(\overline{1}) \text{ and } \omega(i) \neq \overline{\omega(j)} \text{ for } j < i\}$$

This notation makes manipulating the permutations easier and adjusts more easily when n varies. Standard permutation notation is more quickly interpreted for matrix represenations and for dimension attitudes. These notations will be used interchangeably throughout.

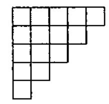


Figure 1.2: ρ_5

Partition representation of a cell of $OGr(n, 2n+1)$

The cells of $OGr(n, 2n+1)$ can also be described by partitions. Let ρ_n be the partition $(n, n-1, \ldots, 2, 1)$ (see Figure 1.2). Let

$$\mathcal{P}_n = \{\lambda = (\lambda_1, \lambda_2, \ldots) \mid \lambda_1 > \lambda_2 > \ldots \text{ and } \lambda \subset \rho_n\}.$$

There is a one-to-one correspondence between elements of W' and elements of $\mathcal{P}_n$. For $\omega \in W'$ and $\lambda \in \mathcal{P}_n$, ω corresponds to λ if $\lambda_i = \max(n+1-\omega(i), 0)$ for $i = 1, \ldots, n$. Equivalently, let $l(\lambda)$ be the number of nonzero rows of λ. Then ω corresponds to λ if

$$\lambda_i = \begin{cases} n+1-\omega(i) & \text{for } i \leq l(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.2.6.

$$\text{Table 1.3} \leftrightarrow \begin{bmatrix} \bullet & 1 & 0 & 0 & 0 \\ \bullet & 0 & * & * & 1 \end{bmatrix} \leftrightarrow \omega = 25314 \equiv \bar{1}20\bar{2}1 \leftrightarrow \lambda = (1) = \square.$$

For the fixed flag $F_. \in OFl(2n+1)$, the Schubert cell of $OGr(n, 2n+1)$ corresponding to the partition $\lambda \in \mathcal{P}_n$ is denoted $\Omega_\lambda(F_.)$.

$$\Omega_\lambda(F_.) = \{V \in OGr(n, 2n+1) \mid j_i = n+1-\lambda_i \text{ for } 1 \leq i \leq l(\lambda) \text{ and } j_i > n+1 \text{ if } i > l(\lambda)\}$$

where $\{j_1 < \cdots < j_n\}$ is the set of jumping numbers for V. The closure of this cell is the Schubert variety $\overline{\Omega_\lambda}(F_.)$ where

$$\overline{\Omega_\lambda}(F_.) = \{V \in OGr(n, 2n+1) \mid \dim(V \cap F_{j_i}) \geq i \quad \forall i\}$$

Lemma 1.2.3. *Let $|\lambda| = \lambda_1 + \lambda_2 + \ldots$. Then the dimension of the cell $\Omega_\lambda(F_.) \subset OGr(n, 2n+1)$ is $\binom{n+1}{2} - |\lambda|$.*

Proof. The $\operatorname{codim}_{OGr(n,2n+1)} \Omega_\lambda(F_.) = |\lambda|$ where $|\lambda| = \lambda_1 + \lambda_2 + \ldots$ by [7]. The cell with maximal codimension is the cell $\Omega_{\rho_n}(F_.) = \{V = F_n\}$ which has dimension zero. This implies that $\dim(OGr(n, 2n+1)) = |\rho_n| = \binom{n+1}{2}$. Thus

$$\dim(\Omega_\lambda(F_.)) = |\rho_n| - |\lambda| = \binom{n+1}{2} - |\lambda|.$$

$\square$

1.2.3 Schubert calculus and cohomology

In the geometric setting, Schubert calculus is normally phrased in cohomological terms (see [6, 7] among others). In the A_n setting, a Schubert variety $\overline{\Omega_\lambda}(F.)$ gives rise to the Schubert cycle

$$[\overline{\Omega_\lambda}(F.)] \in H_*(G(k,n);\mathbb{Z})$$

in homology. This Schubert cycle is independent of the flag $F.$. Its Poincaré dual in cohomology,

$$\sigma_\lambda \in H^*(G(k,n);\mathbb{Z}),$$

is called a *Schubert class.* Now, $H^*(G(k,n);\mathbb{Z})$ is a ring, and the set of Schubert classes $\{\sigma_\lambda\}$ is an additive basis for this ring. In particular, the cup product of two Schubert classes can be written uniquely as a linear combination of basis elements:

$$\sigma_\lambda\sigma_\mu = \sum_\nu c^\nu_{\lambda\mu}\sigma_\nu.$$

The structure coefficients $c^\nu_{\lambda\mu}$ for the cohomology ring $H^*(G(k,n);\mathbb{Z})$ are the Littlewood-Richardson numbers. In a compact manifold such as the Grassmannian, the cup product is Poincaré dual to the intersection product. By the Kleiman-Bertini theorem, for sufficiently general flags $F.$ and $M.$, the Schubert varieties $\overline{\Omega_\lambda}(F.)$ and $\overline{\Omega_\mu}(M.)$ meet transversely, so $\sigma_\lambda\sigma_\mu$ is Poincaré dual to $[\overline{\Omega_\lambda}(F.)\cap\overline{\Omega_\mu}(M.)]$. Now, $\overline{\Omega_\lambda}(F.)\cap\overline{\Omega_\mu}(M.)$ is homologous to a union of other Schubert varieties, and $c^\nu_{\lambda\mu}$ counts the number of components of this union that are Schubert varieties described by ν. Thus, knowing the Littlewood-Richardson numbers answers Question 1.2.2 completely and gives insight into Question 1.2.1. The goal of Schubert calculus is to understand these coefficients from the perspective of the Grassmannian.

More generally, if we have r Schubert conditions and $\overline{\Omega_{\lambda^j}}(F.^j)$ for $1 \le j \le r$ meet properly and transversely, then $\sigma_{\lambda^1}\sigma_{\lambda^2}\dots\sigma_{\lambda^r}$ is Poincaré dual to $[\bigcap_{i=1}^r \overline{\Omega_{\lambda^j}}(F.^j)]$. In particular, $\sigma_{\boxplus}$ (where $\boxplus$ is a $k\times(n-k)$ partition) is Poincaré dual to a point. Thus, if $\sigma_{\lambda^1}\sigma_{\lambda^2}\dots\sigma_{\lambda^r} = a\sigma_{\boxplus}$, then there are exactly a solutions to the Schubert problem for general F^1, F^2, ..., F^r.

Because of the Schubert cell structure in the orthogonal Grassmannian, one can still ask questions in the B_n setting that are similar to questions 1.2.1 and 1.2.2 posed in the type A_n setting.

1.2.4 Schubert calculus in the B_n setting

In the B_n setting, we have a similar situation (see [7]). The Schubert cell $\Omega_\lambda(F.)$ gives rise to the Schubert cycle

$$[\overline{\Omega_\lambda}(F.)] \in H_*(OGr(n, 2n+1); \mathbb{Z})$$

in homology, or by Poincaré duality, the Schubert class

$$\tau_\lambda \in H^*(OGr(n, 2n+1); \mathbb{Z}).$$

For any fixed $F. \in OFl(2n+1)$, the set of Schubert classes,

$$\{\tau_\lambda \in H^*(OGr(n, 2n+1), \mathbb{Z}) \mid \lambda \in \mathcal{P}_n\}$$

form an additive basis for the cohomology of the orthogonal Grassmannian, or equivalently, for its Chow ring[1] [17]. So multiplying two Schubert cycles via the cup product yields a linear combination of Schubert cycles.

$$\tau_\lambda \tau_\mu = \sum_{\nu \in \mathcal{P}_n} a^\nu_{\lambda\mu} \tau_\nu$$

for some integers $a^\nu_{\lambda\mu}$. These integers are the *type B_n Littlewood-Richardson numbers.* Knowing these numbers allows us to understand the ring structure of the Chow ring or cohomology ring. Equivalently, if $F.$ and $M.$ are in general position, then the fundamental class of the intersection of two Schubert varieties, $[\overline{\Omega_\lambda}(F.) \cap \overline{\Omega_\mu}(M.)]$ is $\sum a^\nu_{\lambda\mu}[\overline{\Omega_\nu}(F.)]$. Note that if r flags $F.^1, \ldots, F.^r \in OFl(2n+1)$ are in general position, then the intersection of Schubert varieties, $\cap_{i=1}^r \overline{\Omega_{\lambda_i}}(F.^i)$ is a transverse intersection.

[1] In this dissertation, we use the complex numbers as the base field. It should be noted that more generally, with certain modifications such as replacing the cohomology ring H^{2*} with the Chow ring CH^*, the complex numbers can be replaced with any algebraically closed field of characteristic $\neq 2$.

1.3 Littlewood-Richardson Rules

A combinatorial algorithm for calculating Littlewood-Richardson numbers is called a *Littlewood-Richardson Rule.* In the A_n setting, several such rules are known. There are classic rules such as Pieri's formula and Giambelli's formula as well as rules involving tableaux [6], puzzles [11, 12], and checker games [22] (for a description of the checker games, see also section 1.4).

Pieri's formula is used in the special case of calculating $\sigma_\lambda \sigma_\mu$ where $\mu = (k)$ is the partition consisting of one row of size k. Pieri's formula is

$$\sigma_\lambda \sigma_{(k)} = \sum \sigma_{\lambda'}$$

which is the sum over those λ' that are obtained from λ by adding k boxes with no two in the same column.

Giambelli's formula says we can decompose σ_λ to be written as an expression involving multiplication and addition of σ_k's. Once σ_λ and σ_μ are expressed this way, Pieri's formula can be used to calculate from there. Giambelli's formula is

$$\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leq i,j \leq n-k}$$

In the B_n setting, there is a Pieri-type rule [9] and a Giambelli-type rule [18] for the orthogonal Grassmannian. In addition, one can use symmetric functions to determine the $a^\nu_{\lambda\mu}$. The Hall-Littlewood symmetric functions, or P-polynomials, $P_\lambda(x;t)$, form a basis for the ring of symmetric functions. We can multiply

$$P_\lambda P_\mu = \sum_{\nu \in \mathcal{P}} b^\nu_{\lambda\mu} P_\nu$$

and determine $b^\nu_{\lambda\mu}$ for all $\nu \in \mathcal{P}$. The cup product $\sigma_\lambda \sigma_\mu$ is then a linear combination with coefficients

$$a^\nu_{\lambda\mu} = \begin{cases} b^\nu_{\lambda\mu} & \text{if } \nu \in \mathcal{P}_n \\ 0 & \text{otherwise} \end{cases}$$

[10, 16]. There is a Maple program written by John Stembridge [20] that will write the product $P_\lambda P_\mu$ as a linear combination of P-polynomials.

Buch, Kresch, and Tamvakis have a combinatorial, maximal orthogonal Littlewood-Richardson rule [3] that uses shifted tableaux. There is, however, no known Littlewood-Richardson rule for $OGr(k, 2n+1)$ when $k < n$.

1.4 Vakil's Geometric Littlewood-Richardson Rule

In recent work, Vakil describes the (type A_n) Littlewood-Richardson numbers geometrically via explicit rational degenerations [22]. Given two flags $F.$ and $M.$ in general position with respect to each other, Vakil gives a sequence of rational, codimension one degenerations in the flag manifold that moves one flag until it coincides with the other. The intersection of two Schubert varieties given by λ and μ is studied as the space changes through the sequence of degenerations. At each step, the space either stays as one component or breaks into two components, each with multiplicity one. The number $c^{\nu}_{\lambda\mu}$ is the number of Schubert variety components described by ν that are the result of the degeneration sequence beginning with the intersection of Schubert varieties given by λ and μ. A combinatorial bookkeeping device called a *checker board* and described in [22] encodes the dimensions of intersection at each step of the degeneration.

Vakil describes $c^{\nu}_{\lambda\mu}$ as the number of checker games starting with the configuration $\circ_{\lambda,\mu}\bullet_{init}$ and ending with the configuration $\circ_{\nu}\bullet_{final}$. Given a starting checker configuration, $\circ\bullet_{init}$, there are partitions λ and μ so that $\circ\bullet_{init}$ corresponds to the Schubert calculation, $\sigma_\lambda\sigma_\mu$. Conversely, if $\sigma_\lambda\sigma_\mu \neq 0$, then there is a unique checker configuration $\circ\bullet_{init}$ such that the sum of all possible outcomes of games beginning with this $\circ\bullet_{init}$ configuration corresponds to $\sum c^{\nu}_{\lambda\mu}\sigma_\nu$. We give a summary here of Vakil's work. For a full description, see [22].

1.4.1 Checker game setup

We start by taking two flags in general position with respect to each other, $F.$ and $M.$, and a representative $V \in \Omega_\lambda(F.) \cap \Omega_\mu(M.)$. We then take an $n \times n$ checker board with n black checkers, no two in the same row or column. The columns of the checker board refer to F_1, F_2, $\ldots$, F_n and the rows refer to M_1, M_2, $\ldots$, M_n. The $\dim(F_l \cap M_k)$ is the number of black checkers in positions (i, j) such that $i \leq k$ and $j \leq l$. In other words, $\dim(F_l \cap M_k)$ is the number of black checkers weakly northwest of the (k, l) position. Figure 1.3 shows a $\bullet$-configuration where, for example, $\dim(F_2 \cap M_3) = 2$. In the setup, $F.$ and $M.$ are in general position with respect to each other, so the beginning $\bullet$-configuration, or $\bullet_{init}$, is given by black checkers on the antidiagonal. See figure 1.5.

To complete the setup of a checker game, the white checkers correspond to $V \in G(k, n)$ that meet the flags $F.$ and $M.$ in the prescribed way. If $\sigma_\lambda\sigma_\mu \neq 0$ then there is a unique way to place k white checkers on the $\bullet_{init}$ board such that

F_1	F_2	F_3	F_4	
			•	M_1
•				M_2
	•			M_3
		•		M_4

Figure 1.3: This •-configuration encodes $\dim(M_i \cap F_j)$. The number of black checkers weakly northwest of (i, j) is $\dim(M_i \cap F_j)$.

F_1	F_2	F_3	F_4	
0	0	0	1	M_1
0	0	1	2	M_2
0	1	2	3	M_3
1	2	3	4	M_4

Figure 1.4: $\dim(F_j \cap M_i)$ for $\bullet_{init}$

1. There are no two white checkers in one row or column

2. $\dim(V \cap F_l \cap M_k)$ is the number of white checkers weakly northwest of the (k, l) position.

Example 1.4.1. In $G(2, 4)$, to calculate $\sigma_{\boxminus}\sigma_{\boxminus}$, we first set up the checker configuration $\circ_{\boxminus,\boxminus}\bullet_{init}$. Two flags, $F.$ and $M.$, in the most general position have dimensions of intersection given in Figure 1.4 which corresponds to the $\bullet_{init}$ checker position in Figure 1.5.

The fundamental class dual to $\sigma_{\boxminus}$ has dimensions of intersection given in Table 1.6 and since $\sigma_{\boxminus}$ is independent of the choice of flag, we also have the dimensions of intersection for $V \cap M_i$ in Table 1.7.

Using $\dim(V \cap F_j)$, $\dim(V \cap M_i)$, and $\dim(M_i \cap F_j)$ to determine the general intersection of $V \cap M. \cap F.$, Figure 1.6 is constructed to describe $\dim(V \cap M_i \cap F_j)$. This corresponds to the $\circ_{\boxminus,\boxminus}\bullet_{init}$ checker configuration in Figure 1.7.

			•
		•	
	•		
•			

Figure 1.5: $\bullet_{init}$ for $n = 4$

	F_0	F_1	F_2	F_3	F_4
$\dim(V \cap F_j)$	0	0	1	2	2

Table 1.6: Attitude table for $\Omega_{(1,1)}(F.)$ for Example 1.4.1

	M_0	M_1	M_2	M_3	M_4
$\dim(V \cap M_i)$	0	0	1	2	2

Table 1.7: Attitude table for $\Omega_{(1,1)}(M.)$ for Example 1.4.1

F_1	F_2	F_3	F_4	
0	0	0	0	M_1
0	0	1	1	M_2
0	1	2	2	M_3
0	1	2	2	M_4

Figure 1.6: initial $\dim(V \cap F_j \cap M_i)$ for $\sigma_{(1,1)}\sigma_{(1,1)}$ (Example 1.4.1)

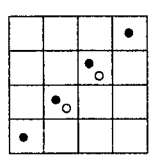


Figure 1.7: $\circ\bullet_{init}$ for $\sigma_{(1,1)}\sigma_{(1,1)}$ (Example 1.4.1)

1.4.2 Playing the game

The flag $M.$ is moved through a series of rational codimension one degenerations, called the *specialization order* until $M.$ coincides with $F.$. At each stage, the white checkers along with the black checkers describe a particular 2-flag Schubert variety. For each degeneration, the space of V that meet $(F., M.)$ in the $\circ\bullet$-way either stays as one component or breaks into two components. Each piece has multiplicity one.

Definition 4.

1. $X_\bullet = \{(F., M.) \in Fl(n) \times Fl(n) |$ $F.$ and $M.$ meet in dimensions described by the $\bullet$-configuration $\}$

2. $X_{\circ\bullet} = \{(V, F., M.) \in G(k,n) \times Fl(n) \times Fl(n) | (F., M.) \in X_\bullet$ and $\dim(V \cap F_j \cap M_i)$ is described by the $\circ\bullet$-configuration. $\}$

3.
$$\begin{aligned}\overline{X}_{\circ\bullet} = \{(V, F., M.) \in G(k,n) \times Fl(n) \times Fl(n) \mid \\ &(F., M.) \in X_\bullet \text{ and } \dim(V \cap F_j \cap M_i) \text{ is \textit{at least} the dimension} \\ &\text{described by the } \circ\bullet\text{-configuration.}\} \\ = Cl_{G(k,n)\times X_\bullet} X_{\circ\bullet}\end{aligned}$$

The black checker moves follow the specialization order which has the property that $\dim X_{\bullet_{next}} = \dim X_\bullet - 1$. On the other hand, at each step, the white checkers move in such a way that the dimension of the fiber of $X_{\circ\bullet} \to X_\bullet$ will be the same as the dimension of the fiber of $X_{\circ\bullet_{next}} \to X_{\bullet_{next}}$. The proof of the geometric Littlewood-Richardson Rule shows that the rules of the game consider exactly the situations where $\dim X_\bullet - 1 = \dim X_{\bullet_{next}}$ but the dimension of the choice of appropriate V remains fixed.

To play a checker game, n black checkers and k white checkers are set up in a prescribed way on an $n \times n$ checkerboard. The black checkers are moved, one swap at a time following the specialization order. This is the order

$$e_{n-1}e_{n-2}\cdots e_2e_1 \quad e_{n-1}\cdots e_2 \quad \cdots \quad e_{n-1}e_{n-2} \quad e_{n-1}$$

	F_0	F_1	F_2	F_3	F_4
$\dim(V \cap F_j)$	0	0	1	1	2

Table 1.8: Attitude table for Example 1.4.2

read right to left where e_i swaps the black checkers in the i^{th} and $(i+1)^{st}$ rows. The two flags, $F.$ and $M.$, begin in the most general position possible, corresponding to $\bullet_{init}$. Figure 1.8 shows the black checker specialization order for an $n = 4$ game. The diagram corresponding to each step of the game shows how $F.$ and $M.$ intersect as $\dim X_\bullet$ decreases by one for each step. The pictures are projectivized, so a point represents 1-dimensional space, a line represents a 2-dimensional space, and the plane represents a 3-dimensional space.

There are specific rules governing the movement of the white checkers at each step (details of the rules can be found in [22]). $\circ\bullet_{init}$ is the initial setup of white and black checkers, $\circ\bullet_{final}$ is the final outcome of a checkergame, and $\circ\bullet_{midsort}$ refers to the configuration of white and black checkers at any intermediate step of the game. In most moves of the checker game, there is a unique way to move the white checkers such that for the projection

$$\pi : G(k,n) \times X_\bullet \to G(k,n)$$

we have $\dim \pi(X_{\circ\bullet}) = \dim \pi(X_{\circ\bullet_{next}})$. In one type of situation there are two possible placements of the white checkers so that $\dim \pi(X_{\circ\bullet}) = \dim \pi(X_{\circ\bullet_{next}})$ (see step 2 of figure 1.10). In this situation, instead of $\circ\bullet$ moving to $\circ\bullet_{next}$, the game splits into two, with configurations $\circ_{stay}\bullet_{next}$ and $\circ_{swap}\bullet_{next}$. Geometrically, as $(F., M.) \in X_\bullet$ specializes to a point in $X_{\bullet_{next}}$, the corresponding fiber $\{V \in G(k,n) | (V, F., M.) \in X_{\circ\bullet}\}$ specializes to a divisor with two components, each having multiplicity one.

Example 1.4.2. A classic example of a Schubert problem asks: How many lines meet four fixed projective lines in $\mathbb{P}^3$? This is equivalent to asking how many $V \in G(2,4)$ meet four fixed two-planes in one-dimensional subspaces.

For one fixed 2-plane, $F_2^{(1)}$, we want dimensions of intersection described in Table 1.8. This corresponds to the partition $\lambda = \square$. To calculate $[\Omega_\square(F.^1) \cap \Omega_\square(F.^2) \cap \Omega_\square(F.^3) \cap \Omega_\square(F.^4)]$ or $\sigma_\square\sigma_\square\sigma_\square\sigma_\square$,we begin with $\sigma_\square\sigma_\square$. Figure 1.9 gives $\dim(V \cap F_j \cap M_i)$ for $1 \leq i,j \leq 4$ and the corresponding $\circ\bullet_{init}$ configuration.

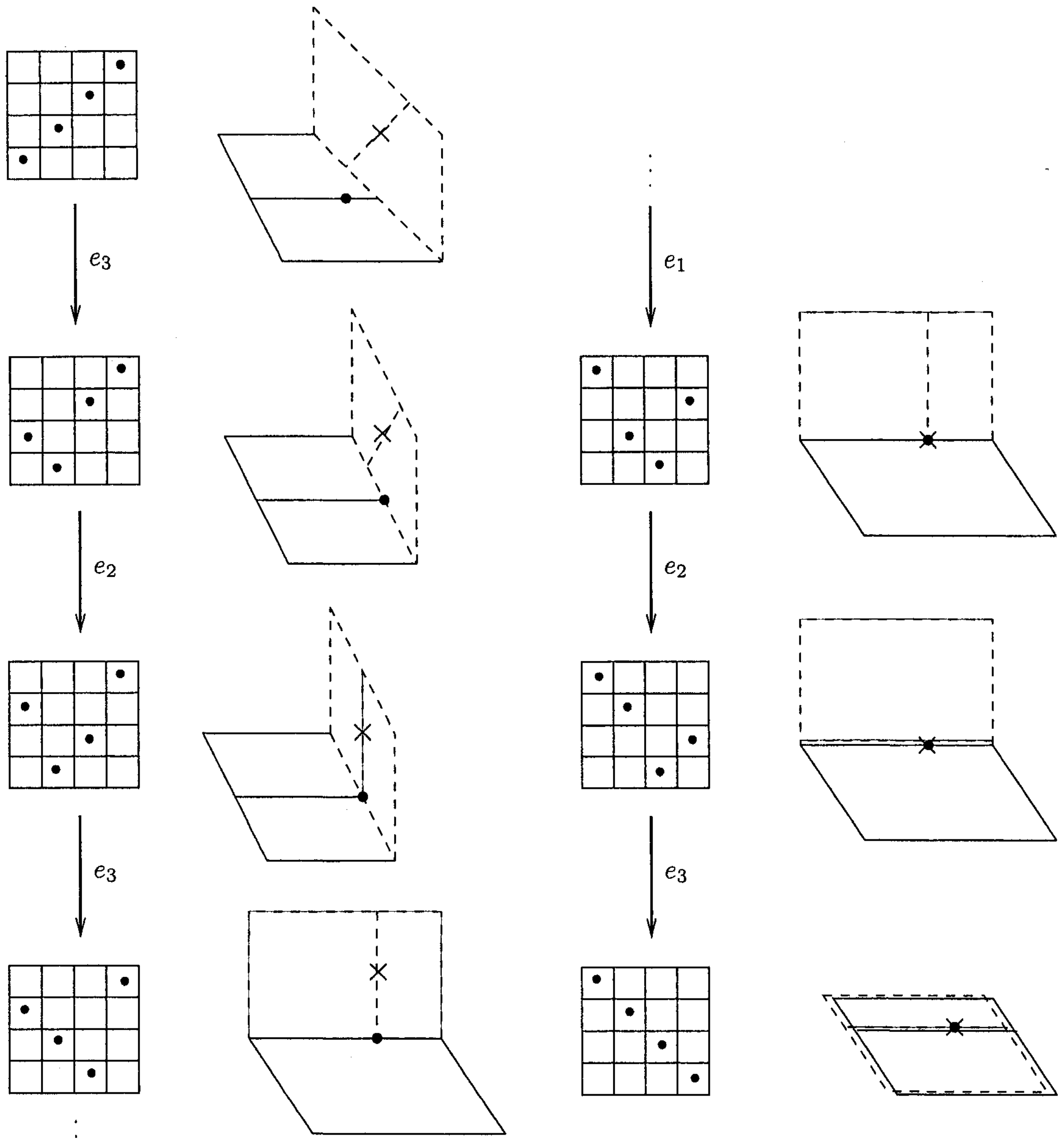


Figure 1.8: Specialization Order for $n = 4$

	F_1	F_2	F_3	F_4
M_1	0	0	0	0
M_2	0	0	0	1
M_3	0	0	0	1
M_4	0	1	1	2

			$\bullet$
		$\bullet$	$\circ$
	$\bullet$		
$\bullet$	$\circ$		

Figure 1.9: Initial $\sigma_{\square}\sigma_{\square}$ configuration (Example 1.4.2)

For this example, we will again consider the projectivized picture, so $V \in G(2,4)$ will be drawn as a dashed line. For $\circ\bullet_{init}$, we can think of V as the span of 2 points described by the two white checkers. One point lies on a general point of $F_2 \cap M_4 = F_2$ (not on F_1). The other point lies on a general point of $F_4 \cap M_2 = M_2$ (not on M_1). Figure 1.10 walks through the $\sigma_{\square}\sigma_{\square}$ checker game and shows the corresponding geometric interpretation. Note that in the initial configuration $\dim(\Omega_{\square}(F_.) \cap \Omega_{\square}(M_.)) = 2$ and $\dim\{V\}$ remains 2 through the game. Note also the split into two components at the second step, which results in $\sigma_{\square}\sigma_{\square} = \sigma_{\yng(2)} + \sigma_{\yng(1,1)}$. Continuing the calculation of $(\sigma_{\square})^4$, we have

$$
\begin{aligned}
\sigma_{\square}\sigma_{\square}\sigma_{\square}\sigma_{\square} &= (\sigma_{\yng(2)} + \sigma_{\yng(1,1)})\sigma_{\square}\sigma_{\square} \\
&= (\sigma_{\yng(2)}\sigma_{\square})\sigma_{\square} + (\sigma_{\yng(1,1)}\sigma_{\square})\sigma_{\square}.
\end{aligned}
$$

Now, two new checker games must be set up and played. Figure 1.11 gives the dimensions and $\circ\bullet_{init}$ configuration for $\sigma_{\yng(2)}\sigma_{\square}$ and Figure 1.12 gives the same for $\sigma_{\yng(1,1)}\sigma_{\square}$. Both games, $\sigma_{\yng(2)}\sigma_{\square}$ and $\sigma_{\yng(1,1)}\sigma_{\square}$, yield a $\circ\bullet_{final}$ configuration corresponding to $\yng(2,1)$, so the last game is the same for both branches, namely $\sigma_{\yng(2,1)}\sigma_{\square}$. The $\dim(V \cap F_j \cap M_i)$ and $\circ\bullet_{init}$ configuration for $\sigma_{\yng(2,1)}\sigma_{\square}$ are given in Figure 1.13.

The outcome of this final game is $\sigma_{\yng(2,1)}\sigma_{\square} = \sigma_{\yng(2,2)}$, so $(\sigma_{\square})^4 = 2\sigma_{\yng(2,2)}$, meaning there are two solutions to the original Schubert problem.

1.4.3 The geometric Littlewood-Richardson rule

Consider the following commutative diagram:

$$
\begin{tikzcd}
\overline{X}_{\circ\bullet} \arrow[r, "\text{open}"'] \arrow[d] & \overline{X}_{\circ\bullet} \arrow[d] & D_X \arrow[l, "\text{closed}"'] \arrow[d] \\
X_{\bullet} \arrow[r, "\text{open}"'] & X_{\bullet} \cup X_{\bullet_{next}} & X_{\bullet_{next}} \arrow[l, "\text{closed}"']
\end{tikzcd}
\tag{1.3}
$$

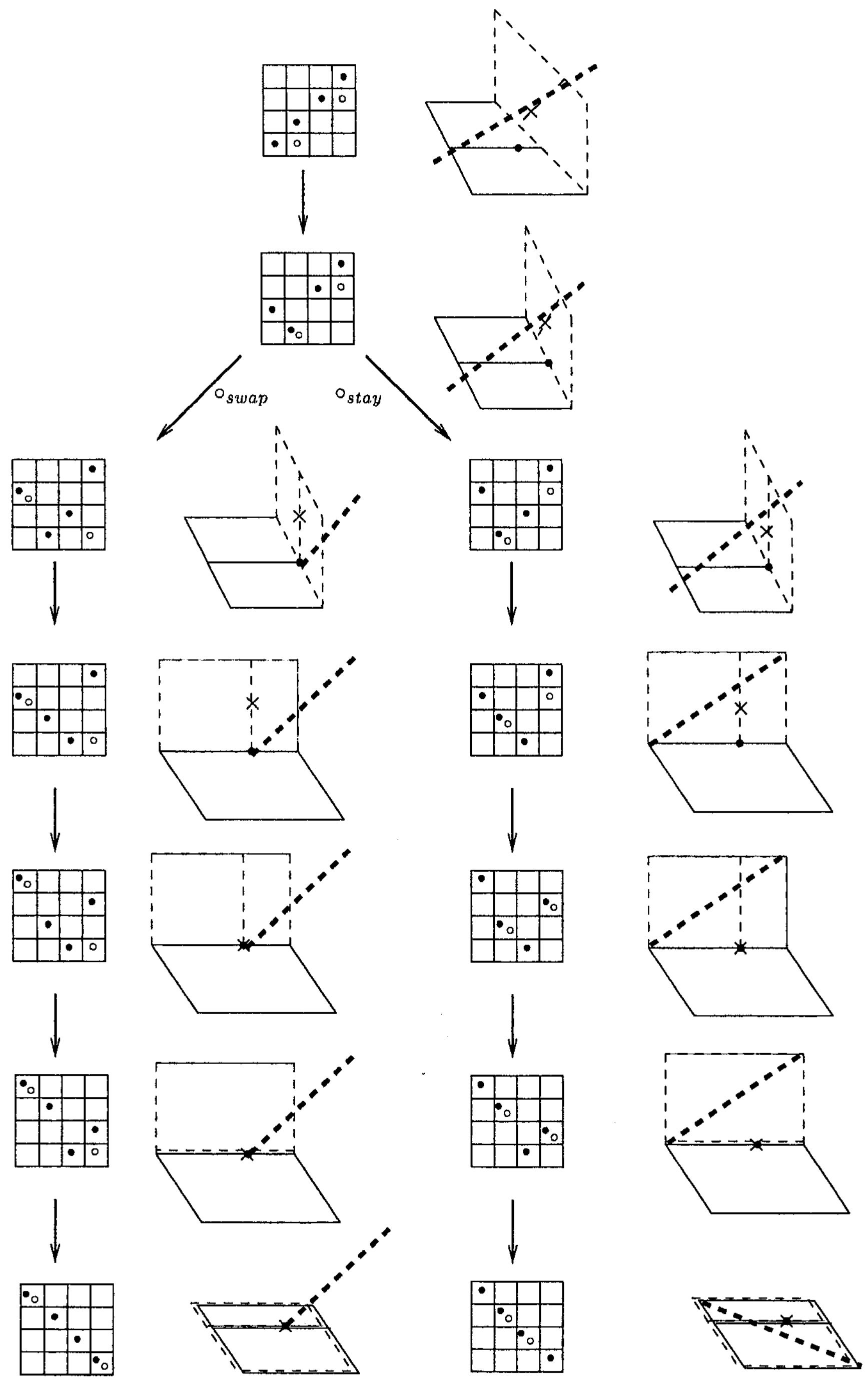


Figure 1.10: $\sigma_{\square}\sigma_{\square} = \sigma_{(2)} + \sigma_{(1,1)}$

F_1	F_2	F_3	F_4	
0	0	0	0	M_1
0	0	0	1	M_2
0	0	0	1	M_3
1	1	1	2	M_4

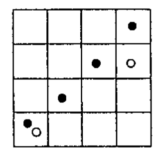


Figure 1.11: Initial $\sigma_{(2)}\sigma_{\Box}$ configuration (Example 1.4.2)

F_1	F_2	F_3	F_4	
0	0	0	0	M_1
0	0	1	1	M_2
0	0	1	1	M_3
0	1	2	2	M_4

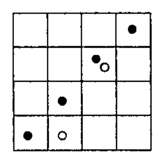


Figure 1.12: Initial $\sigma_{(1,1)}\sigma_{\Box}$ configuration (Example 1.4.2)

F_1	F_2	F_3	F_4	
0	0	0	0	M_1
0	0	1	1	M_2
0	0	1	1	M_3
1	1	2	2	M_4

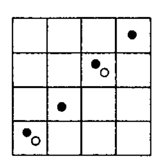


Figure 1.13: Initial $\sigma_{(2,1)}\sigma_{\Box}$ configuration (Example 1.4.2)

The closure of the upper left corner is taken in $G(k,n) \times X_\bullet$ while the closure of the upper middle is taken in $G(k,n) \times (X_\bullet \cup X_{\bullet_{next}})$. Define D_X as the fiber product for the right square of diagram (1.3). The Geometric Littlewood-Richardson rule states that at each stage of a checker game, the space of all V that meet $(F_., M_.)$ in the $\circ\bullet_{next}$ way is exactly D_X. In other words, the rules of the checker game were made in the right way, causing no additional nor lost components and no extra multiplicities. Precisely,

Theorem 1.4.1 (Geometric Littlewood-Richardson Rule).
$D_X = \overline{X}_{\circ_{stay}\bullet_{next}}, \overline{X}_{\circ_{swap}\bullet_{next}}, \text{ or } \overline{X}_{\circ_{stay}\bullet_{next}} \cup \overline{X}_{\circ_{swap}\bullet_{next}}$

The proof of the Geometric Littlewood-Richardson Rule first shows that the components of D_X are exactly those described by each step of the checker game, namely $\overline{X}_{\circ_{stay}\bullet_{next}}$, $\overline{X}_{\circ_{swap}\bullet_{next}}$, or the union of the two. The proof then shows that each of these components has multiplicity 1.

1.4.4 The importance of a *geometric* rule

Why do we want a geometric interpretation of the Littlewood-Richardson Rule? The initial question posed is really a geometric question: Given two Schubert conditions and two general flags, we want to find out about the set of $V \in G(k,n)$ that satisfy the given conditions with respect to the given flags.

Before the work of Vakil, geometers were forced to analyze general Schubert problems algebraically. In particular, two Schubert conditions and two general flags correspond to two Schur polynomials (already we've jumped into algebra). Using a (non-geometric) Littlewood-Richardson rule, the product of these two Schur polynomials gives a sum of various Schur polynomials. These new Schur polynomials correspond to a sum of Schubert conditions. By the Kleiman-Bertini theorem, this new sum of Schubert varieties is homologous to the original intersection. This amounts to a black box approach to determining the intersection of two Schubert classes.

On the other hand, Vakil's geometric rule gives an explicit step-by-step description of how the intersection and the sum are connected. This is more direct and in general preferable to a black box explanation. More importantly, this proof opens the door for other applications. In particular, Schubert induction [23] is a consequence of this geometric proof. Among other things, Schubert induction shows that when the intersection of a collection of Schubert varieties has dimension zero, then for most choices of real flags, all of the expected solutions are real. Vakil's geometric proof using degeneration methods will likely also lead to numerical solutions of general Schubert problems. In applications such as control theory, the existence of real solutions and numerical algorithms are tremendously important, so the geometric Littlewood-Richardson rule promises to have a significant impact on such fields.

1.5 Schubert Induction

In this section, we give a brief introduction to Schubert Induction, an important application of Vakil's geometric Littlewood-Richardson rule. For a complete proof and discussion, see [23].

Consider Example 1.4.2 of four fixed projective lines in $\mathbb{P}^3$. The question of how many other projective lines intersect the four fixed lines can be answered by the Geometric Littlewood-Richardson Rule (of course it can be answered other ways as well). Suppose we want both of the solutions to be real. We might ask: Is there a choice of four real flags such that the intersection of the corresponding Schubert varieties yields all real solutions? The answer is yes, and Schubert Induction is used to prove this.

Let P be a property of morphisms that depends only on dense open subsets of the range, meaning $f : X \to Z$ has property P if there is a dense open subset $V \subset Z$ such that $f|_{f^{-1}(V)}$ has the property P. For a given $\bullet$-configuration and subvariety $Y \subset G(k,n) \times X_\bullet$, let $\pi_{\bullet,Y}$ be the projection

$$\begin{array}{c}\{(V, F_\cdot^1, \ldots, F_\cdot^m) \in G(k,n) \times Fl(n)^{m-2} \times X_\bullet \mid V \in \overline{\Omega}_{\alpha_i}(F_\cdot^i) \text{ for } 1 \leq i \leq m-2 \text{ and } (V, F_\cdot^{m-1}, F_\cdot^m) \in Y\} \\ \big\downarrow{\scriptstyle \pi_{\bullet,Y}} \\ Fl(n)^{m-2} \times X_\bullet\end{array}$$

Theorem 1.5.1 (Schubert Induction Theorem).

IF

1. $\overline{\Omega}_\alpha \subset G(k,n) \times Fl(n)$ *is the universal Schubert variety,* $\overline{\Omega}_\alpha = \{(V, F_\cdot) | V \in \overline{\Omega}_\alpha(F_\cdot)\}$*, then the projection* $S : \overline{\Omega}_\alpha \to Fl(n)$ *has property* P *for all partitions* α
2. *whenever* $\pi_{\bullet_{next}, D_X}$ *has* P*, then* $\pi_{\bullet, \overline{X}_{\circ\bullet}}$ *has* P
3. *whenever the checker game splits and both* $\pi_{\bullet_{next}, \overline{X}_{\circ_{stay}\bullet_{next}}}$ *and* $\pi_{\bullet_{next}, \overline{X}_{\circ_{swap}\bullet_{next}}}$ *have* P *then* $\pi_{\bullet_{next}, D_X}$ *has* P

THEN

We can have many Schubert conditions and property P *will still hold. In other words, property* P *holds* $\forall\ m, \alpha_1, \ldots, \alpha_m$ *for the projection*

$$\begin{array}{c}\{(V, F_\cdot^1, \ldots, F_\cdot^m) \in G(k,n) \times Fl(n)^m | V \in \overline{\Omega}_{\alpha_i}(F_\cdot^i) \text{ for } 1 \leq i \leq m\} \\ \big\downarrow \\ Fl(n)^m.\end{array} \tag{1.4}$$

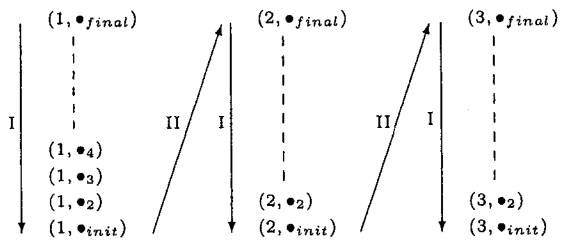


Figure 1.14: Order of induction using Type I and Type II induction steps

The proof is a backwards induction on the steps through the specialization order (type I induction step) and a forward induction on the number of Schubert conditions, m (type II induction step). Details can be found in [23].

1. Type I: this type of induction step says if we are at $\bullet_i \neq \bullet_{final}$ and we know that at $\bullet_{i+1}$ the projection onto $Fl(n)^m$ has P, then at $\bullet_i$ the projection also has property P.
2. Type II: this induction step says if we are at a $\bullet_{init}$ position and we have m Schubert conditions, then we can rewrite these conditons as the final step of a game played using $m + 1$ Schubert conditions.

The order of induction can be pictured as in Figure 1.14. Let $(m, \bullet_j)$ be an ordered pair where m is the number of Schubert conditions and $\bullet_j$ is the j^{th} step in the specialization order, with $\bullet_{init} = \bullet_1$.

1.5.1 Applications

Schubert Induction can be applied to several questions. Two are summarized here.

Reality

Returning to the question of reality, are Schubert problems enumerative over the reals?

Corollary 1.5.2. *There is a dense open subset of $Fl(n)^m$, all real, such that appropriately asked Schubert questions on these flags will yield all real solutions.*

Example 1.5.1. To see how Schubert Induction helps answer this question, we look at the example in $G(2,4)$ explored earlier (example 1.4.2). The calculations can be summarized by the tree diagram on the right side of Figure 1.15. Each vertical line represents the movement through

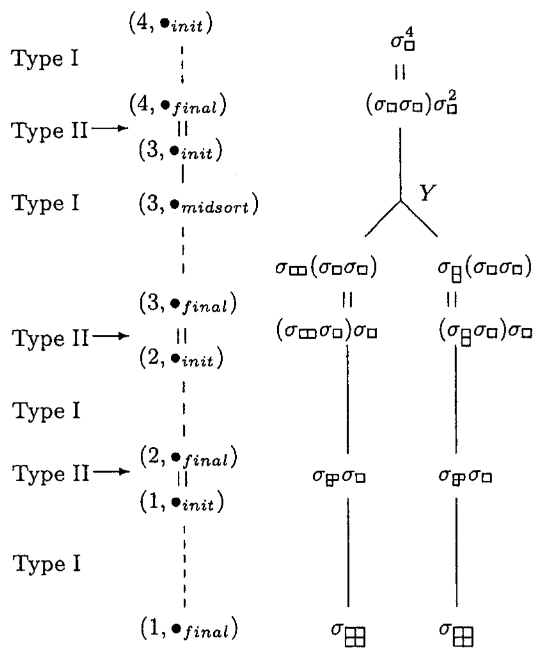


Figure 1.15: Working from bottom to top, Schubert Induction is used to show this Schubert problem is enumerative over the reals.

the specialization order, or playing the checker game. The branching at Y is where the checker game $\sigma_{\square}\sigma_{\square}$ splits into two games.

First we check that $S : \Omega_\alpha(F.) \to Fl(4)$ has property P for all partitions α. In the reality question, we are checking that for all partions α there is a dense open real subset U of $Fl(4)$ such that $S^{-1}(U) \subset \Omega_\alpha(F.)(\mathbb{R})$. First note that the only single Schubert Condition that yields a finite number of solutions is $\sigma_{\boxplus}$. So we find a dense open subset of $Fl(4)(\mathbb{R})$ such that its preimage is real. $\sigma_{\boxplus}$ is Poincaré dual to $\{V \in G(k,n) | V = F_2\}$. Choose $U = Fl(4)(\mathbb{R})$. Condition 1 is met. Conditions 2 and 3 are also met by the choice of U.

Following Figure 1.15 we can walk through the induction order beginning at both leaves on the bottom and simultaneously moving up. At Y, each component has property P, so by condition 3 of the theorem, when combining $\circ_{swap}$ and $\circ_{stay}$ we will still have property P.

Generic Smoothness

We know that a Schubert problem over an algebraically closed field of characteristic zero will always give the expected number or dimension of solutions. For an algebraically closed field of positive characteristic, p, this is not always the case since there do exist nowhere smooth, surjective maps between smooth spaces. Consider the following definition:

Definition 5. A morphism $f : X \to Y$ is *generically smooth* if there is a dense open subset $V \subset Y$ and a dense open subset $U \subset f^{-1}(V)$ such that f is smooth on U.

If the projection in (1.4) is generically smooth, then we have the following corollary:

Corollary 1.5.3. *If a particular dimension of solutions or a certain finite number of solutions are expected over $\mathbb{C}$, then the same dimension or number of solutions can be expected over any algebraically closed field. In particular, this holds for fields of characteristic $p > 0$.*

A traditional way to show that Schubert calculus is enumerative over $\mathbb{C}$ is to use the Kleiman-Bertini Theorem. Kleiman-Bertini implies that the projection in (1.4) is generically smooth over $\mathbb{C}$, which in turn shows that Schubert calculus is enumerative over $\mathbb{C}$. Unfortunately, Kleiman-Bertini fails in characteristic $p > 0$, so until [22], there was no way to show that the projection in (1.4) is generically smooth over a field of characteristic $p > 0$. Generic smoothness used as property P in Theorem 1.5.1 satisfies all three hypotheses, so by Schubert Induction, (1.4) is generically smooth. This implies corollary 1.5.3.

1.6 Dissertation goals and realities

The goal of this project is to state and prove a type B_n geometric Littlewood-Richardson rule for the maximal orthogonal Grassmannian along the lines of Vakil's type A_n rule. A strategy similar to [22] is used here, however the number of cases to be considered is significantly larger. There are three times as many trivial cases and at least six times as many non-trivial cases.

This dissertation is not a complete proof of a type B_n geometric Littlewood-Richardson rule for the maximal orthogonal Grassmannian (conjecture 3). We present a substantial portion of the proof of conjecture 3 including a specialization order for the degeneration of one orthogonal flag into another (section 2.2), preliminary lemmas with careful consideration to the effect of the bilinear form on the geometry (in particular, lemma 3.3.1), definitions of spaces needed for the proof of the main conjecture, and proofs for certain cases in the conjecture. We give complete proofs for the trivial cases, both the s_0 and s_i cases (section 4.2). We leave for future work nontrivial cases where there is a white checker in column $c+1$. For nontrivial cases with no white checker in column $c+1$, we give partial results for the case of nontrivial s_i moves where there is no white checker in row $d_E < n$. We give complete proofs in the cases

1. Nontrivial s_i-moves where there is a white checker in row $d_E < n$.

2. Nontrivial case for s_0 moves.

Chapter 2

STATEMENT OF THE RULE

	F_1	F_2	F_3	F_4	F_5
M_1	0	0	0	0	1
M_2	0	1	1	1	2
M_3	0	1	2	2	3
M_4	0	1	2	3	4
M_5	1	2	3	4	5

$\longleftrightarrow$

				•
	•			
		•		
			•	
•				

Figure 2.1: The dimensions of $M_i \cap F_j$ encoded by the black checker configuration on the right are listed on the left.

	F_1	F_2	F_3	F_4	F_5
M_1	0	0	0	0	1
M_2	0	0	0	1	2
M_3	0	0	1	2	3
M_4	0	1	2	3	4
M_5	1	2	3	4	5

$\longleftrightarrow$

				•
			•	
		•		
	•			
•				

Figure 2.2: $\bullet_{init}$-configuration for $n = 2$

In this chapter, we state the type B_n geometric Littlewood-Richardson rule and describe some preliminaries needed for the proof.

2.1 Black Checker Diagrams and double Schubert cells

Consider a $(2n+1) \times (2n+1)$ checker board with $2n+1$ black checkers, no two in the same row or column. We make a rank table describing $\dim(M_i \cap F_j)$ such that $\dim(M_i \cap F_j)$ is the number of black checkers weakly northwest of position (i, j). We define

$$X_\bullet = \{(M_., F_.) \in OFl(2n+1) \times OFl(2n+1) \mid M_. \text{ and } F_. \text{ meet in dimensions}$$
$$\text{described by the } \bullet\text{-configuration}\}$$

$X_\bullet$ is an example of a double Schubert cell. See figure 2.1.

The dimensions of two two transverse flags are encoded by black checkers in positions $(i, 2n+2-i)$ for $1 \leq i \leq 2n+1$. We will call this configuration $\bullet_{init}$, the *inital* black checker configuration. See figure 2.2.

The configuration of black checkers in positions (i, i) for $1 \leq i \leq 2n+1$ describes the diagonal of $OFl(2n+1) \times OFl(2n+1)$. Such a configuration will be called $\bullet_{final}$, the *final* black checker configuration. The corresponding double Schubert cell is $X_{\bullet_{final}}$.

2.2 Specialization order

Given an isotropic flag $F.$ and an isotropic flag $M.$ that is transverse to $F.$, we give a sequence of n^2 rational curves in $OFl(2n+1)$, each of degree one or two. The sequence of curves moves the flag $M.$ until it coincides with $F.$. Traveling along each curve causes a minimal increase in intersection between $M.$ and $F.$, a codimension one degeneration. A degeneration corresponds to moving black checkers on the $(2n+1) \times (2n+1)$ checkerboard. The prescribed sequence of black checker moves is called the *specialization order.*

Theorem 2.2.1. *There is a sequence of n^2 codimension one degenerations taking an arbitrary isotropic flag to a fixed isotropic flag. Each degeneration respects isotropy and corresponds to a curve of degree one or two in the orthogonal flag variety .*

Consider the subset S of the symmetric group on $2n+1$ elements.

$$S = \{s_0, s_1, \ldots, s_{n-1}\} \subset S_{2n+1}$$

where $s_0 = (n, n+2)$ and $s_i = (n+1+i, n+2+i)(n+1-i, n-i)$ for $1 \leq i \leq n-1$. S generates W, the Weyl group for B_n, see equation (1.1). Note that

1. $s_i^2 = 1$
2. $(s_i s_{i+1})^3 = 1$ for $1 \leq i \leq n-2$
3. $(s_0 s_1)^4 = 1$
4. $(s_i s_j)^2 = 1$ for $|i-j| \geq 2$.

So (W, S) is a Coxeter system [5]. The specialization order comes from a walk through the Bruhat order of W, beginning with a representation of the longest word ω_0 (length n^2) and ending with the identity, 1. Let

$$\omega_0 = (s_{n-1}s_{n-2}\ldots s_0 \ldots s_{n-2}s_{n-1})(s_{n-2}s_{n-3}\ldots s_0 \ldots s_{n-3}s_{n-2})\ldots(s_1 s_0 s_1)(s_0). \tag{2.1}$$

If $s_j \in S$ is the rightmost letter of a word ω, apply the permutation s_j to ω, giving $\omega' = \omega s_j$. By the properties of the Coxeter system (W, S), we have $s_j^2 = 1$ so ω' has length 1 less than ω. The representation of ω_0 in equation (2.1) (reading right to left) gives the specialization order for the deformation of $M.$ into $F.$.

An s_0 move swaps the black checkers in rows n and $n+2$. An $s_i, i \neq 0$, move swaps the black checkers in rows $n+1+i$ and $n+2+i$ and simultaneously swaps the checkers in rows $n+1-i$ and $n-i$. An s_i move swaps four checkers while an s_0 move swaps only two. See figure 2.3. The movement of checkers occurs with some symmetry about position $(n+1, n+1)$. In this choice of path, the radius of movement of checkers about position $(n+1, n+1)$ increases as the specialization order progresses.

Example 2.2.1. For $n = 3$, $\omega_0 = s_2 s_1 s_0 s_1 s_2 s_1 s_0 s_1 s_0$. The intermediate checker configurations correspond to partial factorizations of ω_0. Figure 2.3 shows the nine moves of the black checkers and the corresponding partial factorizations and permutations. The Weyl group for $n = 3$ can be realized as a polytope. Consider the polytope obtained by shaving off the corners and edges of a cube. The faces of the cube become octagons, the edges become squares, and the vertices become hexagons. Such a polytope is called a permutahedron of type B_3 [5]. See figure 2.4. The vertices correspond to elements of the Weyl group. The edges between squares and octagons correspond to s_0 moves. The edges between hexagons and octagons correspond to s_1 moves. And the edges between squares and hexagons correspond to s_2 moves. If we place the longest word at the north pole of the permutahedron, then the identity, 1, is the south pole. The specialization order is a prescribed path along the edges (with the first step along the edge between the square and the octagon) to move from the north pole to the south pole.

We now describe the degeneration at each step by describing an explicit rational curve that the $M_.$ flag follows to move one step closer to becoming the $F_.$ flag. Given any $F_.$ and $M_.$ in $\bullet$-position, we can choose a basis such that $F_.$ is the standard flag and $M_.$ is given by the $\bullet$-configuration. Specifically, $F_j = \langle e_1, \ldots, e_j \rangle$ and $M_i = \langle e_{j_1}, \ldots, e_{j_i} \rangle$ where j_k is the column of the black checker in row k.

For a fixed $F_.$, we describe a curve in $X_\bullet \cup X_{\bullet_{next}}$ as the set of points $(F_., M_.^p)$ for $p = [s, t] \in \mathbb{P}^1$ with the properties:

1. $M_.^{[0,1]} = M_.$
2. $(F_., M_.^p) \in X_\bullet$ for $p \neq [1, 0]$
3. $(F_., M_.^{[1,0]}) \in X_{\bullet_{next}}$

The degeneration for an s_i move is linear while the degeneration for an s_0 move is quadratic.

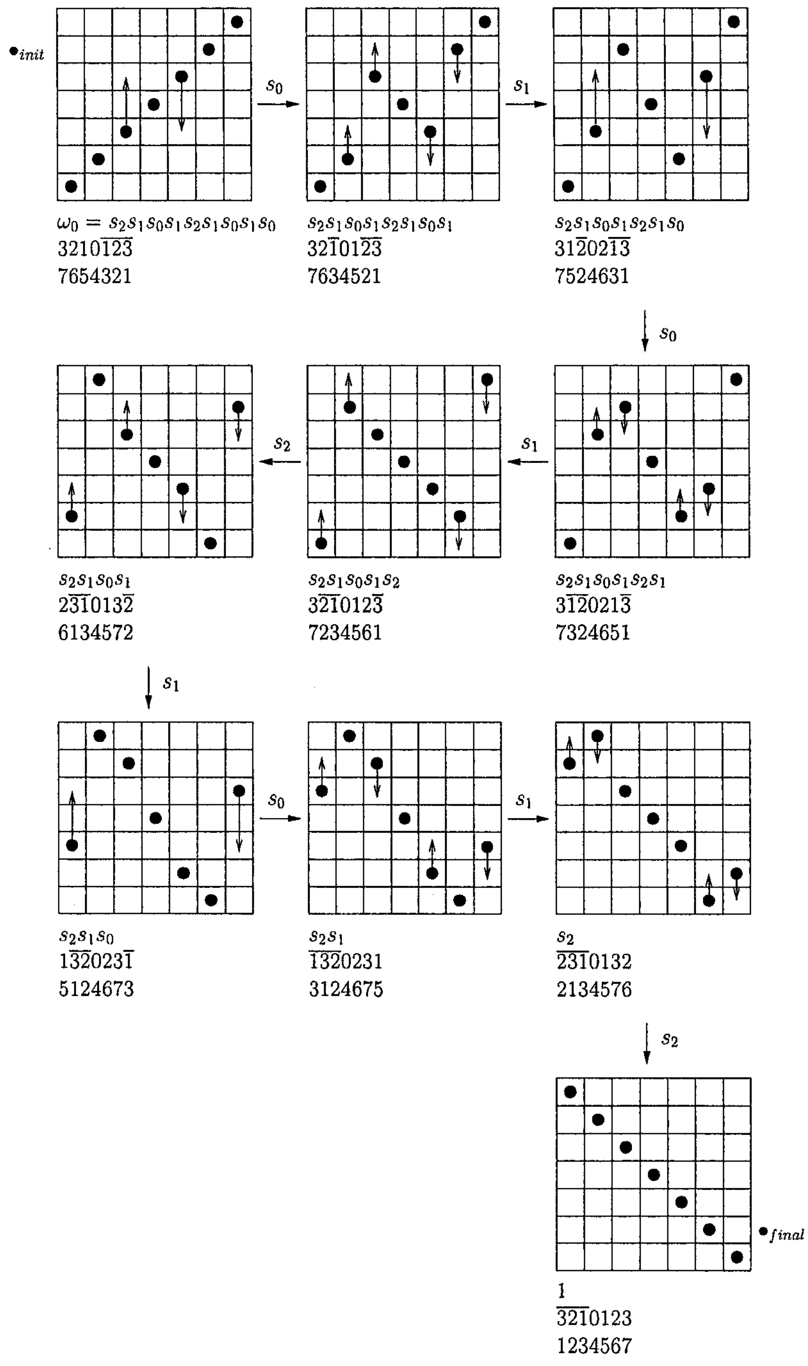


Figure 2.3: Specialization order for B_3. Each step is labeled with the corresponding element of the Weyl group, a permutation of $\{\bar{3},\bar{2},\bar{1},0,1,2,3\}$, and a permutation of $\{1,2,3,4,5,6,7\}$

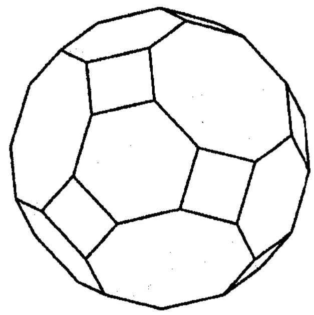


Figure 2.4: Permutahedron of type B_3 from [5]

2.2.1 Rational curve for an s_i move

In an s_i move, $i \neq 0$, black checkers in rows $n+1-i$ and $n-i$ swap and black checkers in rows $n+1+i$ and $n+2+i$ swap. While there are four checkers moving, only two spaces, M_{n-i} and $M_{n-i}^{\perp} = M_{n+1+i}$ are actually moving in the degeneration. Since $M_{n-i}^{\perp}$ is determined by M_{n-i}, we can describe the curve by showing what happens to the flag

$$M_{\cdot}^{p} = (M_1 \subsetneq \cdots \subsetneq M_{n-i}^{p} \subsetneq M_{n+1-i} \subsetneq \cdots \subsetneq M_n)$$

For all $p \in \mathbb{P}^1$, we need $M_{n-i}^{p} \subset M_{n+1-i}$. Define $M_{\cdot}^{p}$ as

$$M_k^p = M_k = \langle e_{j_1}, \ldots, e_{j_k} \rangle$$

for $1 \leq k \leq n-i-1$ and for $n+1-i \leq k \leq n$ and

$$M_{n-i}^{p} = M_{n-i-1} + \langle s e_{j_{n+1-i}} + t e_{j_{n-i}} \rangle$$

Note that for any choice $p \in \mathbb{P}^1$, M_{n-i}^{p} is isotropic and $M_{n-i}^{p} \subset M_{n+1-i}$.

Example 2.2.2. We illustrate this linear curve with the s_1 move ($n=3$) shown in figure 2.5. For this configuration, we choose a basis for $F_{\cdot}$ and $M_{\cdot}$ so that $F_{\cdot}$ is the standard flag with $F_j = \langle e_1, \ldots, e_j \rangle$ and

$$M_1 = \langle e_7 \rangle$$
$$M_2 = \langle e_7, e_6 \rangle$$
$$M_3 = \langle e_7, e_6, e_3 \rangle$$

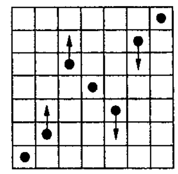


Figure 2.5: Black checker configuration for an s_1 move

Let $M_2^p = \langle e_7, se_3 + te_6 \rangle$. When $p = [0,1]$, then $M_2^p = M_2$ and $M^p = M_\cdot$. Consider a matrix whose first i rows span M_i^p (as long as $s \neq t$).

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & s & 0 & 0 & t & 0 \\ 0 & 0 & t & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & -t & 0 & 0 \\ 0 & t & 0 & 0 & s & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In row 5, we use $-t$ instead of t so that $M_5^p = (M_2^p)^\perp$. If $p = [0,1]$, then we have nonzero entries in $\bullet$-positions and zeroes otherwise. If $p = [1,0]$, then $M_2^p = \langle e_7, e_3 \rangle$ and $(F_\cdot, M^p) \in X_{\bullet_{next}}$. To make a path to $p = [1,0]$, we begin by moving away from the point $[0,1]$ and so without loss of generality, let $s = 1$. For our example, this gives the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & t & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 & 0 \\ 0 & t & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first i rows should span M_i^p, so again without loss of generality, we can change row 3 and row 6 giving the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then for any choice of t (even $t = 1$) the rows span as expected. The span of the rows of this matrix describe a linear path from a general point in $X_\bullet \cup X_{\bullet_{next}}$ to a general point in $X_{\bullet_{next}}$.

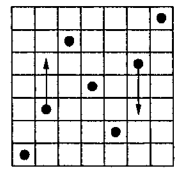


Figure 2.6: Black checker configuration for an s_0 move

2.2.2 Rational curve for an s_0 move

For an s_0 move, black checkers in rows n and $n+2$ swap positions. Here, two checkers are moving and two spaces, M_n^p and $M_{n+1}^p = (M_n^p)^\perp$ are moving in the degeneration. Since M_{n+1}^p is determined by M_n^p, we describe the curve by considering $M_\cdot^p = (M_1 \subsetneq \cdots \subsetneq M_{n-1} \subsetneq M_n^p)$. We need M_n^p isotropic and $M_n^p \subset M_{n+2}$. Define $M_\cdot^p$ by

$$M_k^p = \langle e_{j_1}, \ldots, e_{j_k} \rangle$$

for $1 \leq k \leq n-1$ and

$$M_n^p = M_{n-1} + \langle 2s^2 e_{j_{n+2}} + 2tse_{n+1} - t^2 e_{j_n} \rangle$$

Note that $j_{n+1} = n+1$ since the center checker is in position $(n+1, n+1)$.

For any choice $p = [s,t] \in \mathbb{P}^1$, M_n^p is isotropic and quadratic terms are needed to satisfy isotropy.

Example 2.2.3. We illustrate this quadratic curve with the $\bullet$-configuration shown in figure 2.6 whose next move is of type s_0 ($n=3$). From this $\bullet$-configuration, we choose a basis for $F_\cdot$ and $M_\cdot$. $F_\cdot$ is the standard flag and

$$M_1 = \langle e_7 \rangle$$
$$M_2 = \langle e_7, e_3 \rangle$$
$$M_3 = \langle e_7, e_3, e_6 \rangle$$
$$M_4 = \langle e_7, e_3, e_6, e_4 \rangle = M_3^\perp$$
$$M_5 = \langle e_7, e_3, e_6, e_4, e_2 \rangle = M_2^\perp$$

Let $M_\cdot^p = (M_1 \subsetneq M_2 \subsetneq M_3^p)$ with M_4, M_5, M_6 and M_7 determined as perps.

$$M_3^p = \langle e_7, e_3, 2s^2 e_2 + 2tse_4 - t^2 e_6 \rangle$$

When $p = [0, 1]$,

$$M_3^p = \langle e_7, e_3, -e_6 \rangle = \langle e_7, e_3, e_6 \rangle = M_3$$

which gives $M^p_\cdot = M_\cdot$.

Consider a matrix whose first i rows span M_i^p (as long as $s \neq t$).

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2s^2 & 0 & 2ts & 0 & -t^2 & 0 \\ 0 & 2ts & 0 & (t^2 - s^2) & 0 & ts & 0 \\ 0 & -t^2 & 0 & 2ts & 0 & 2s^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If $p = [0, 1]$ then we have nonzero entries in $\bullet$-positions and zeros otherwise. If $p = [1, 0]$, then $M_3^p = \langle e_7, e_3, e_2 \rangle$ which puts $(F_\cdot, M^p_\cdot) \in X_{\bullet_{next}}$. To make a path to $p = [1, 0]$, we begin by moving away from the point $[0, 1]$ and so without loss of generality, let $s = 1$. For our example, this gives the matrix

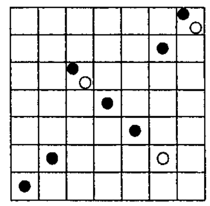
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2t & 0 & -t^2 & 0 \\ 0 & 2t & 0 & (t^2 - 1) & 0 & t & 0 \\ 0 & -t^2 & 0 & 2t & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And since the first i rows should span M_i^p, we can change row 5 to give a new matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2t & 0 & -t^2 & 0 \\ 0 & 2t & 0 & (t^2 - 1) & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and then for any choice of t (even $t = \sqrt{-2}$) the rows span as expected. The span of the rows of this matrix describe a degree two path from a general point in $X_\bullet \cup X_{\bullet_{next}}$ to a general point in $X_{\bullet_{next}}$.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
M_1	0	0	0	0	0	0	1
M_2	0	0	0	0	0	1	2
M_3	0	0	1	1	1	2	3
M_4	0	0	1	2	2	3	4
M_5	0	0	1	2	3	4	5
M_6	0	1	2	3	4	5	6
M_7	1	2	3	4	5	6	7



	F_1	F_2	F_3	F_4	F_5	F_6	F_7
M_1	0	0	0	0	0	0	1
M_2	0	0	0	0	0	0	1
M_3	0	0	1	1	1	1	2
M_4	0	0	1	1	1	1	2
M_5	0	0	1	1	1	1	2
M_6	0	0	1	1	1	2	3
M_7	0	0	1	1	1	2	3

Figure 2.7: The black checkers encode the dimensions of $M_i \cap F_j$ listed on the left and the white checkers encode the dimensions of $V \cap M_i \cap F_j$ listed on the right.

2.3 White checkers and two flag Schubert varieties

In addition to the black checkers, we place n white checkers on the board. These white checkers encode the dimensions $\dim(V \cap M_i \cap F_j)$ for $F.$ and $M.$ in $\bullet$-position and $V \in OGr(n, 2n+1)$. The number of white checkers weakly northwest of position (i, j) is $\dim(V \cap M_i \cap F_j)$. Note that the southernmost row then encodes $\dim(V \cap F_j)$ and the easternmost column encodes $\dim(V \cap M_i)$. See figure 2.7 for an example.

Definition 6. A white checker configuration $\circ$ is *happy with respect to* $\bullet$ if each white checker has one black checker weakly north of it in the same column and one black checker weakly west of it in the same row (this is Vakil's definition of happy). When there is no confusion over which $\bullet$-configuration is meant, then we just say *happy*.

Definition 7. A white checker configuration $\circ$ is *pairwise happy* if it is happy and if there is a white checker in position (r_i, c_i) and a white checker in position (r_j, c_j), then $r_i + r_j \neq 2n+2$ and $c_i + c_j \neq 2n+2$ for any $1 \leq i, j \leq n$ (even $i = j$).

Definition 8. A white checker configuration $\circ$ is *isotropically happy* if it is pairwise happy and there exists some $V \in OGr(n, 2n+1)$ that meets the flags in exactly the way described by the $\circ\bullet$ configuration. In other words, there cannot be a more specialized white checker configuration that describes the full set of isotropic V's that meet the flags in the way described by the original configuration.

2.3.1 Initial configurations

For an intersection of two Schubert varieties (with respect to transverse isotropic flags, i.e. flags in the $\bullet_{init}$-position), we give a construction to place white checkers on the initial black checker board. We will call this white checker configuration $\circ_{init}$.

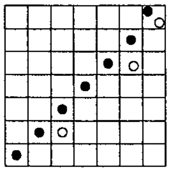


Figure 2.8: For $n = 3$, $\tau_{(1)}\tau_{(3,1)}$ corresponds to this initial position.

Theorem 2.3.1. *There is a one-to-one correspondence between initial white checker configurations of type $\circ_{init}$ and non-empty type B_n two-flag Schubert varieties.*

For a Schubert question posed as $\Omega_\lambda(F.) \cap \Omega_\mu(M.)$, we describe how to set up the initial white checker configuration. Labeling columns and rows with $\overline{n}, \overline{n-1}, \ldots, \overline{1}, 0, 1, 2, \ldots, n-1, n$ (recall this notation introduced at the end of section 1.2.2), the jumping numbers for $\lambda \subset \rho_n$ are $j_1 < j_2 < \cdots < j_n$. Let $\ell(\lambda)$ be the number of rows in the partition λ and λ_k be the number of boxes in row k. Then $\lambda^\vee \subset \rho_n$ is the strict partition whose parts complement the parts of λ in the set $\{1, 2, \ldots, n\}$. For $1 \leq k \leq \ell(\lambda)$, we have

$$j_k = \overline{\lambda_k}$$

and for $1 \leq k \leq \ell(\lambda^\vee)$, we have

$$j_{\ell(\lambda)+k} = \lambda^\vee_{\ell(\lambda^\vee)+1-k}$$

The jumping numbers for $\mu \subset \rho_n$ are $i_1 < i_2 < \cdots < i_n$ and are described similarly.

Definition 9. On a $\bullet_{init}$-configuration, define $\circ_{init}$ as the *initial white checker configuration* for $\Omega_\lambda(F.) \cap \Omega_\mu(M.)$. $\circ_{init}$ is constructed by placing white checkers in positions (i_k, j_{n+1-k}) for $1 \leq k \leq n$.

See figure 2.8 for an example $\circ_{init}$-configuration. We make some observations about $\circ_{init}$.

Lemma 2.3.1. *The white checker configuration $\circ_{init}$ has no white checker in the center column and no white checker in the center row. Equivalently, $0 \notin \{j_1, j_2, \ldots, j_n\}$ and $0 \notin \{i_1, i_2, \ldots, i_n\}$.*

Proof. This is clear by construction of the jumping numbers from λ and μ. We can also see this by constructing the matrix representation for V. Suppose $0 \in \{j_1, ..., j_n\}$. Then the matrix representation of V has a row v_α of the form

$$\begin{bmatrix} * & * & \dots & * & 1 & 0 & \dots & 0 \end{bmatrix}$$

where the 1 is in the middle column. This requires that $0 = B(v_\alpha, v_\alpha) = 1$, a contradiction, in order for V to be isotropic. A similar argument works for $\{i_1, i_2, ..., i_n\}$. □

Lemma 2.3.2. *The set* $\{\overline{k}, k\} \not\subset \{j_1, ..., j_n\}$ *and* $\{\overline{k}, k\} \not\subset \{i_1, ..., i_n\}$ *for any* $0 \leq k \leq n$.

Proof. This is again clear by construction of the jumping numbers from λ and μ. We can also see this by constructing the matrix representation for V. Suppose $\{\overline{k}, k\} \subset \{j_1, j_2, ..., j_n\}$. Then V is represented by an $n \times (2n+1)$ matrix of the form

$$V = \begin{bmatrix} \vdots & & & & & & & & & & & \\ * & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & & \\ * & \dots & * & * & * & \dots & * & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & & \end{bmatrix}$$

with 1's in columns $\overline{k}$ and k. But then V is not isotropic because the rows v_α and v_β (listed in the matrix for V) give $B(v_\alpha, v_\beta) = 1$. A similar argument works for $\{i_1, i_2, ..., i_n\}$. □

Lemma 2.3.3. *If* μ *is not contained in* $\lambda^\vee$ *then the initial white checker configuration* $\circ_{init}$ *is not happy and* $\Omega_\lambda(F_.) \cap \Omega_\mu(M_.) = \emptyset$.

Proof. (See also [3] defining $f(\lambda, \mu; \nu)$.) In the $\bullet_{init}$ configuration, the $2n+1$ black checkers are in positions $(k, \overline{k})$ for $0 \leq k \leq n$. As a consequence, if a $\circ_{init}$ configuration is happy, then $r + c \geq 0$ for a white checker in position (r, c). If $\mu \not\subset \lambda^\vee$ then either

1. $\ell(\mu) > \ell(\lambda^\vee)$ or
2. $\ell(\mu) \leq \ell(\lambda^\vee)$ and there is some k, $1 \leq k \leq \ell(\mu)$ such that $\mu_k > \lambda^\vee_k$.

Case 1: $\ell(\mu) > \ell(\lambda^\vee)$

In this case there is a white checker in position $(i_{\ell(\mu)}, j_{n+1-\ell(\mu)})$ where

$$i_{\ell(\mu)} = \overline{\mu_{\ell(\mu)}} < 0$$

and

$$\begin{aligned} j_{n+1-\ell(\mu)} &< j_{n+1-\ell(\lambda^\vee)} \\ &= j_{n+1-(n-\ell(\lambda))} \\ &= j_{\ell(\lambda)+1} \end{aligned}$$

Now, $m = \ell(\lambda) + 1$ is the minimal m such that $j_m > 0$, so $j_{n+1-\ell(\mu)}$ must be negative. This implies $i_{\ell(\mu)} + j_{n+1-\ell(\mu)} < 0$ and thus this white checker position is not happy.

Case 2: $\ell(\mu) \leq \ell(\lambda^\vee)$ and there is some k, $1 \leq k \leq \ell(\mu)$ such that $\mu_k > \lambda_k^\vee$.

Note that for $1 \leq m \leq \ell(\mu)$, the white checker positions are $(\overline{\mu_m}, \lambda_m^\vee)$. Consider the white checker in position $(\overline{\mu_k}, \lambda_k^\vee)$. Checking the necessary happiness condition, $\overline{\mu_k} + \lambda_k^\vee < 0$ since $\mu_k > \lambda_k^\vee$. So this white checker position is not happy. □

Lemma 2.3.4. *On a $\bullet_{init}$ checkerboard, the $\circ_{init}$ configuration is the least specialized $\circ$-configuration that still describes the required intersections for $\Omega_\lambda(F.) \cap \Omega_\mu(M.)$.*

Proof. Suppose $\circ \neq \circ_{init}$ is an isotropically happy white checker configuration on a $\bullet_{init}$ checkerboard such that for $(F., M.) \in X_{\bullet_{init}}$ there exists $V \in \Omega_\lambda(F.) \cap \Omega_\mu(M.)$. Let the positions of the n white checkers in the $\circ$-configuration be described by pairs $(i_{\sigma(k)}, j_k)$ for $1 \leq k \leq n$ and $\sigma \in S_n$. Let k be the smallest number of $\{2, ..., n\}$ such that $\sigma(k) > \sigma(k-1)$ (i.e. $i_{\sigma(k)} > i_{\sigma(k-1)}$).

In order to satisfy the jumps in dimension for $V \cap F_j$ and $V \cap M_i$, any good initial white checker configuration $\circ$ must have one white checker in each of rows $i_1, ..., i_n$ and in each of columns $j_1, ..., j_n$. The claim to be proved is: $\circ_{init}$ is the least specialized of these possible configurations. We first look at the case where $k = 2$.

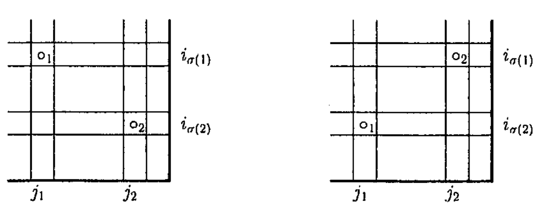


Figure 2.9: $k = 2$

If we swap checkers $\circ_1$ and $\circ_2$, we have a less specialized configuration because in figure 2.9 on the left side, $\dim(V \cap F_{j_2} \cap M_{i_{\sigma(2)}}) = 2$ and $\dim(V \cap F_{j_1} \cap M_{i_{\sigma(1)}}) = 1$. In figure 2.9 on

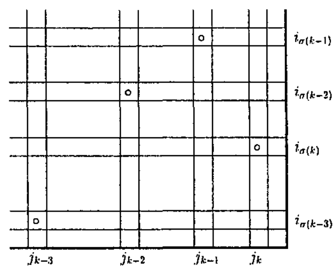


Figure 2.10: $k > 2$

the right side, $\dim(V \cap F_{j_2} \cap M_{i_{\sigma(2)}}) = 2$ and $\dim(V \cap F_{j_1} \cap M_{i_{\sigma(1)}}) = 0$ while still preserving dimension jumps at $j_1, j_2, i_{\sigma(1)}$, and $i_{\sigma(2)}$.

For the case $k > 2$, we have figure 2.10. Notice for $\alpha < k$ that $\dim(V \cap F_{j_\alpha} \cap M_{i_{\sigma(\alpha)}}) = 1$ but for k, $\dim(V \cap F_{j_k} \cap M_{i_{\sigma(\alpha)}}) \geq 2$. Thus we can swap the checkers in columns j_k and j_{k-1}. This decreases the dimension of intersection, but does not change the dimension jumps. If $i_{\sigma(k-1)} > i_{\sigma(k-2)}$ now, then swap these two checkers. Continue until all white checkers are creating a positive slope (i.e. rows decrease west to east). This is the least specialized configuration, and still preserves dimension jumps. Now the checkers are in positions (i_{n+1-k}, j_k). □

Lemma 2.3.5. *If $\Omega_\lambda(F.) \cap \Omega_\mu(M.) \neq \emptyset$ then the corresponding $\circ_{init}$ configuration is isotropically happy.*

Proof. Suppose $\circ_{init}$ does not yield a happy checker configuration. Then there must be a white checker in position (i_{n+1-k}, j_k) such that $i_{n+1-k} + j_k < 0$. Since $\circ_{init}$ is the least specialized position for V, there is no way to decrease $\dim(V \cap F_{j_k} \cap M_{i_{n+1-k}})$. Because a white checker is located at (i_{n+1-k}, j_k), we have $\dim(V \cap F_{j_k} \cap M_{i_{n+1-k}}) \geq 1$. But according to the $\bullet_{init}$ configuration, we have $\dim(F_{j_k} \cap M_{i_{n+1-k}}) = 0$, a contradiction. So $\Omega_\lambda(M.) \cap \Omega_\mu(F.) = \emptyset$.

By construction of $\circ_{init}$, if $\circ_{init}$ is happy, then it is pairwise happy.

It remains to be shown that $\circ_{init}$ is isotropically happy. □

2.3.2 Midsort

Definition 10. A checker diagram is described as *midsort* if the black and white checkers are positioned in such a way that the black checkers are in one of the specialization order configurations and the white checkers are in positions that follow from prescribed moves beginning with a $\circ_{init}$-configuration.

We observe the following characteristic of midsort configurations:

Labeling columns (and rows) with $1, 2, \ldots, 2n{+}1$, let $(c{+}1)$ be the column of the westernmost rising black checker. Denote

$$\underline{x} = 2n + 1 - x. \tag{2.2}$$

Then $\underline{c} = 2n + 1 - c$ is the column of the easternmost descending black checker.

White checkers in columns $1 \leq col \leq c$ tend in the same direction as the black checkers. And for white checkers in columns $\underline{c} \leq col \leq 2n + 1$, a similar trend occurs. We state the following as conjectures. These and other characteristics of midsort will be proven by induction when analysis of all cases is complete.

Conjecture 1. *In a midsort checker diagram, white checkers in columns $1 \leq col \leq c$ and columns $\underline{c} \leq col \leq 2n + 1$ decrease in rows from west to east.*

Conjecture 2. *In a midsort checker diagram, white checkers in columns $c < col < \underline{c}$ increase in rows from west to east.*

2.3.3 Reading the final answer

A final $\circ\bullet$-configuration has black checkers in positions (k, k) for $\overline{n} \leq k \leq n$. The n white checkers are in positions along the same diagonal. We determine ν for $\Omega_\nu(F_.)$ by recording the positions of the white checkers in columns (or equivalently rows) $\overline{n} \leq col < 0$. Call these positions $\alpha_1 < \alpha_2 < \cdots < \alpha_{\ell(\nu)} < 0$. Then $\nu = (\overline{\alpha_1}, \overline{\alpha_2}, \ldots, \overline{\alpha_{\ell(\nu)}})$.

2.4 Statement of Rule

2.4.1 Geometric Statement

Conjecture 3. *Let τ_λ and τ_μ be Schubert classes for the orthogonal Grassmannian $OGr(n, 2n+1)$, with $\tau_\lambda \tau_\mu = \sum a^\nu_{\lambda\mu} \tau_\nu$. Then the coefficient $a^\nu_{\lambda\mu}$ is equal to the number of isotropic checker games with input λ, μ and output ν.*

2.4.2 Combinatorial Statement

Definition 11. Let d_E be the row of the easternmost descending black checker and let d_W be the row of the western descending black checker. For s_i moves, either $1 \leq d_E < n$ or $n+2 \leq d_E < 2n+1$. Row d_W has the property that $d_W = 2n+1-d_E$. For s_0 moves, $d_E = n$ and there is no row d_W.

Definition 12. Define c as the rightmost column with a black checker in position $(2n+2-c, c) = (\underline{c}, c)$. In other words, c is the column of the rightmost $(c < n)$ black checker on the antidiagonal. Then $\underline{c} > n+2$ (recall (2.2)) is the column of the rightmost descending black checker.

The rules for moving white checkers are as follows:

For an s_0 move, there is either a white checker in row n or in row $n+2$, but not both. If there is not a white checker in row n, then we call this the trivial case and the white checkers stay. If there is a white checker in row n, the row of the descending black checker, then we consider columns $n+2$ through $\underline{c+1}$. See Figure 2.11. If there is a white checker in one of these columns, we choose the top most white checker. The location of this white checker and the white checker in row n determine if these checkers stay, swap, or stay and swap (in a split). In the split possibility, the pair of checkers can stay, or if there are no white checkers in the rectangle between them, they can swap. A white checker in the rectangle is called a *blocker.* See Figure 2.12 for an example. Table 2.1 summarizes the s_0 white checker moves (when there is no white checker in column $c+1$).

In any s_i move, if there is no white checker in row d_E and no white checker in row d_W, then we call this a trivial move and the white checkers stay.

In an s_i move with a white checker in row $d_E < n$, we consider columns $d_E + 1$ through $\underline{c+1}$. See Figure 2.13. If there is a white checker in one of these columns, we choose the top most

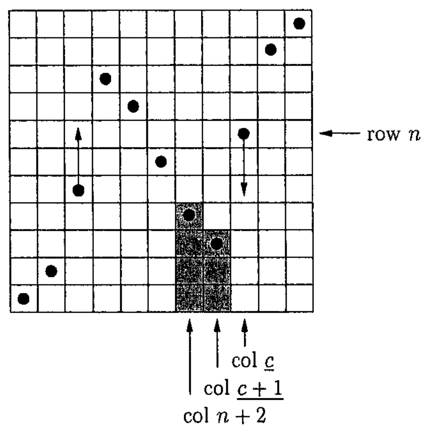


Figure 2.11: If there is a white checker in row n in an s_0 move, then look for the northermost white checker in the highlighted region ($n+2 \leq col \leq \underline{c+1}$).

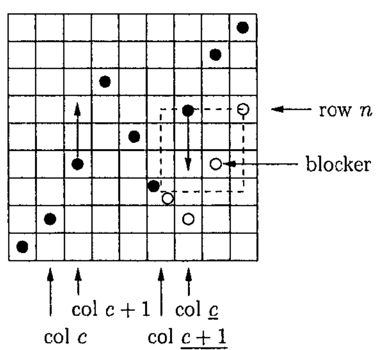


Figure 2.12: Example of a blocker in an s_0 move.

		Is there a white checker in row n?		
		Yes, in $col = \underline{c}$	Yes, in $col > \underline{c}$	No
Top WC in column	Yes	swap	swap if no blocker *or* stay	stay
$n+2 \leq col \leq \underline{c+1}$?	No	stay	stay	stay

Table 2.1: White checker moves for the s_0 case when there is no white checker in column $c+1$

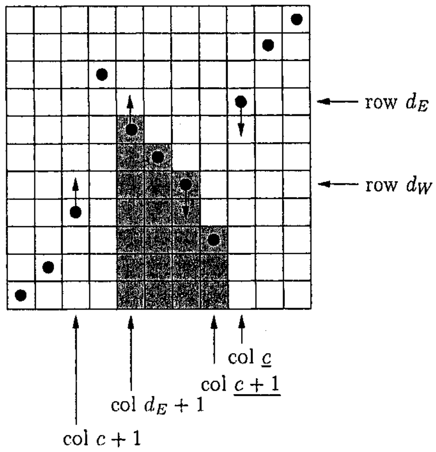


Figure 2.13: If there is a white checker in row $d_E < n$ in an s_i move, then look for the northermost white checker in the highlighted region ($d_E + 1 \leq col \leq \underline{c+1}$).

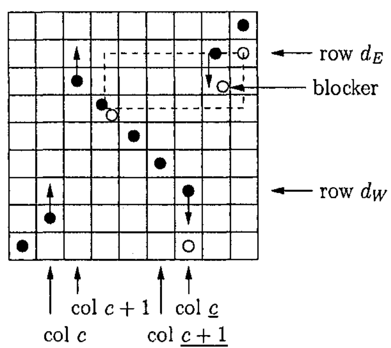


Figure 2.14: Example of a blocker in an s_i move.

white checker. The location of this white checker and the white checker in row d_E determine if the move is a stay, a swap, or a split (separate stay and a swap branches). In the split case, the pair of white checkers can stay, or if there is not a blocker, the two white checkers can swap. See Figure 2.14 for an example of a blocker. Table 2.2 summarizes the s_i white checker moves for a white checker in row $d_E < n$ (when there is no white checker in column $c + 1$).

		Where is the white checker in row $d_E < n$?	
		In $col = \underline{c}$	In $col > \underline{c}$
Top WC in column $d_E + 1 \leq col \leq \underline{c+1}$?	Yes in rising checker square	swap	swap
	Yes, elsewhere	swap	swap if no blocker *or* stay
	No	stay	stay

Table 2.2: White checker moves for the s_i case with a white checker in row $d_E < n$ and when there is no white checker in column $c + 1$

2.5 Cleanups

After a particular move is completed, the white checkers may be unhappy (or isotropically unhappy) with respect to the new $\bullet$ configuration.

2.5.1 Regular cleanups

A white checker is not be happy if there is no longer a black checker either weakly north or weakly west of the white checker.

After an s_i move, at most two white checkers may be unhappy. After an s_0 move, at most one white checker may be unhappy. In the clean-up phase, if a white checker is not happy, then move it either up or left until it becomes happy. This is always possible, in a unique way. Cleanups across the midline involve one checker. Cleanups not crossing the midline involve two checkers.

Recall that a white checker is not pairwise happy if it is in row $n+1$ or column $n+1$. Therefore, a white checker in row $n+2$ that moves up to become pairwise happy must move to row n (not row $n+1$). Similarly, cleanups moving left across column $n+1$ will never stop in column $n+1$.

A white checker moving from row $r \neq n+2$ to row $r-k$ in a cleanup wil have a corresponding white checker in row $2n+2-(r-k)$ that moves to row $2n+2-r$. This paired move always occurs in the maximal case because there is no white checker in row $r-k$ implying that there is a white checker in row $2n+2-(r-k)$. To preserve pairwise happiness, we must make this a paired move. Similarly, we also have paired moves when cleaning up white checkers by shifting left.

Examples of cleanups occur in the A_n case [22]. In the B_n case, we see examples of paired and unpaired cleanups even in very small examples.

Example 2.5.1. With $n = 2$, consider the first two moves shown in Figure 2.15 in the game $\lambda = (2,1)$ and $\mu = \emptyset$. The first move is a trivial s_0 move (see Section 2.4.2 for combinatorial rules on moving white checkers). The cleanup raises one white checker across row $n+1$. The second move is a trivial s_1 move. The cleanup move shifts two white checkers up one row each.

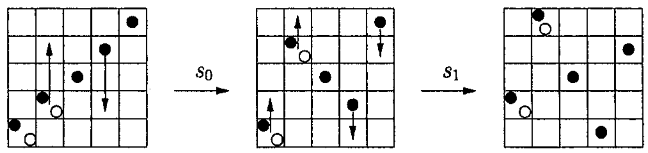


Figure 2.15: The first two moves in an $n = 2$ game with $\lambda = (2, 1)$ and $\mu = \emptyset$. Both s_0 and s_1 moves are stay moves that require cleanups.

2.5.2 isotropic cleanups

Given a white checker configuration, we want to determine if the configuration is isotropically happy and if not, determine the least specialized $\circ$-configuration that is isotropically happy. It is an open question to find combinatorial rules to determine if a $\circ$-configuration is isotropicially happy.

On the other hand, any particular $\circ\bullet$-configuration can be determined to be isotropically happy or not by asking about ideal membership in an ideal generated by quadrics. This can, in theory, be solved algorithmically by Gröbner basis methods.

Chapter 3

PRELIMINARY LEMMAS

3.1 Orthogonal Bott-Samelson varieties

3.1.1 Bott-Samelson varieties

In the proof of a Geometric Littlewood-Richardson Rule, Vakil adds a third row to (1.3) which considers a *Bott-Samelson variety* instead of the Grassmann variety. We recall for the reader (nearly verbatim) the definition of the *Bott-Samelson variety* $BS(Q)$ given in [22].

Associate a variety to the following data, $(Q, \dim, n)$.

1. Q is a finite subset of the plane, with the partial order $\prec$ given by domination ($\mathbf{a} \prec \mathbf{b}$ if $\mathbf{a}$ is weakly northwest of $\mathbf{b}$). We require Q to have a maximum element max and a minimum element min. (We visualize the plane so that moving south corresponds to increasing the first coordinate and moving east corresponds to increasing the second coordinate, in keeping with the labeling convention for tables.)

2. $\dim : Q \to \{0, 1, 2, \ldots, n\}$ is an order preserving map, denoted *dimension.*

3. If $[\mathbf{a}, \mathbf{b}]$ is a covering relation in Q (i.e. minimal interval: $\mathbf{a} \prec \mathbf{b}$, and there is no $\mathbf{c} \in Q$ such that $\mathbf{a} \prec \mathbf{c} \prec \mathbf{b}$), then we require that $\dim \mathbf{a} = \dim \mathbf{b} - 1$.

4. If straight *edges* are drawn corresponding to the covering relations, then we require the interior of the graph to be a union of *quadrilaterals*, with four elements of Q as vertices, and four edges of Q as boundary.

We call this data a *quilt*, and abuse notation by denoting it by Q and leaving dim implicit.

Note that the poset Q must be a lattice, i.e. any two elements $\mathbf{x}, \mathbf{y}$ have a unique minimal element dominating both (denoted $\sup(\mathbf{x}, \mathbf{y})$), and a unique maximal element dominated by both (denoted $\inf(\mathbf{x}, \mathbf{y})$). An element of Q at (i, j) is said to be on the *southwest border* (resp. *northeast border*) if there are no other elements (i', j') of Q such that $i' > i$ and $j' < j$ (resp. $i' < i$ and $j' > j$). Thus every element on the boundary of Q is on the southwest border or the northeast border. The maximum and minimum elements are on both.

Definition 13. Let K be an algebraically closed field with char $K \neq 2$. Define the *Bott-Samelson variety* $BS(Q)$ associated to a quilt Q to be the variety parameterizing a $(\dim \mathbf{a})$-plane V_a in K^n for each $\mathbf{a} \in Q$, with $V_a \subset V_b$ for $\mathbf{a} \prec \mathbf{b}$.

$BS(Q)$ is a smooth [22], closed subvariety of $\prod_{a \in Q} Gr(\dim a, n)$. Elements $\mathbf{m}$ of Q will be written in bold font while corresponding vector spaces will be denoted V_m.

Any set S of quadrilaterals of a quilt determines a *stratum* of the Bott-Samelson variety. The closed stratum corresponds to requiring the spaces of the northeast and southwest vertices of each quadrilateral in S to be the same. The open stratum corresponds to also requiring the spaces of the northeast and southwest vertices of each quadrilateral *not* in S to be distinct. Denote the open stratum by $BS(Q)_S$, so the dense open stratum is $BS(Q)_\emptyset$. The open strata give a stratification, the closed strata are smooth, and $\operatorname{codim}_{BS(Q)} BS(Q)_S = |S|$. We depict a stratum by placing an "=" in the quadrilaterals of S, indicating the pairs of spaces that are required to be equal.

3.1.2 Orthogonal Bott-Samelson varieties

Consider a quilt $(Q, \dim, 2n+1)$ with $\dim \max \leq n$. The *orthogonal Bott-Samelson variety* $OBS(Q)$ is a subvariety of the corresponding Bott-Samelson variety $BS(Q)$ defined as

$$OBS(Q) = \{V_\cdot \in BS(Q) \mid V_{max} \text{ is isotropic } \}$$

Lemma 3.1.1. *$OBS(Q)$ is a smooth, closed subvariety of $BS(Q_\circ)$, hence of* $\prod_{\mathbf{m} \in Q} Gr(\dim \mathbf{m}, 2n+1)$

Proof. We have a smooth projection

$$BS(Q) \to G(\dim(\max), 2n+1)$$

And we have a smooth, closed subvariety

$$OGr(\dim(\max), 2n+1) \hookrightarrow G(\dim(\max), 2n+1)$$

And $OBS(Q)$ is the pullback:

$$\begin{tikzcd} OBS(Q) \arrow[r] \arrow[d] & BS(Q) \arrow[d] \\ OGr(\dim(\max), 2n+1) \arrow[r, hook, "\text{closed}"] & G(\dim(\max), 2n+1) \end{tikzcd}$$

This implies that $OBS(Q)$ is smooth, $OBS(Q)$ is a closed subvariety of $BS(Q)$, and $OBS(Q)$ inherits a stratification from $BS(Q)$. □

Example 3.1.1 (Quilts generated by a set of white checkers.). Given a white checker configuration, define the associated quilt $Q_\circ$ by including the squares of the checker board where there is a white checker weakly north and a white checker weakly west of the square. Include a "zero element" $\mathbf{0}$ northwest of the white checkers. For $\mathbf{s} \in Q$, let $\dim \mathbf{s}$ be the number of white checkers $\mathbf{s}$ dominates, so $\dim \mathbf{0} = 0$, and $\dim \mathbf{s}$ is the edge-distance from $\mathbf{s}$ to $\mathbf{0}$. If we allow V_{max} to vary in $G(\dim \max, 2n+1)$ then we have $BS(Q_\circ)$, and if we additionally require the maximal space to be isotropic, then we have $OBS(Q_\circ)$. See (among others) figures 4.3 and 4.4.

3.2 General results about vector spaces

Here we collect some general technical lemmas regarding vector spaces. These results will be referenced in the proof of conjecture 3.

3.2.1 Results for any vector spaces

Lemma 3.2.1. *Let M_S, M_B, V_S, and V_B be vector spaces such that $M_S \subset M_B$, and $V_S \subset V_B$. And let $\ell = \dim(V_B \cap M_S) - \dim(V_S \cap M_S)$ and $\ell' = \dim(V_B \cap M_B) - \dim(V_S \cap M_B)$. Then $\ell' \geq \ell$.*

Proof. Since $M_S \subset M_B$, we have

$$\dim(V_S \cap M_B) - \dim(V_S \cap M_S) = \alpha \geq 0$$

and

$$\dim(V_B \cap M_B) - \dim(V_B \cap M_S) = \beta \geq 0$$

Since $V_S \subset V_B$, the increase in dimension from $V_B \cap M_S$ to $V_B \cap M_B$ must be at least α. So $\beta \geq \alpha$. Now,

$$\begin{aligned} \ell' - \ell &= \dim(V_B \cap M_B) - \dim(V_S \cap M_B) - (\dim(V_B \cap M_S) - \dim(V_S \cap M_S)) \\ &= \beta - \alpha \\ &\geq 0 \end{aligned}$$

□

Lemma 3.2.2. *Let V and W be vector spaces (not necessarily isotropic). Then*

(a) $(V + W)^\perp = V^\perp \cap W^\perp$

(b) $(V \cap W)^\perp = V^\perp + W^\perp$

Proof of part (a). If $a \in (V + W)^\perp$ then $B(a, b) = 0$ for all $b \in V + W$. Since $V \subset V + W$, we know $B(a, v) = 0$ for all $v \in V$, so $a \in V^\perp$. Similarly, $a \in W^\perp$. So $v \in V^\perp \cap W^\perp$.

Conversely, if $a \in V^\perp \cap W^\perp$ then $B(a, v) = 0$ for all $v \in V$ and $B(a, w) = 0$ for all $w \in W$. Now, every element $b \in V + W$ can be written as $b = v + w$ for some $v \in V$ and $w \in W$. Then

$$B(a, b) = B(a, v + w) = B(a, v) + B(a, w) = 0$$

So $a \in (V + W)^\perp$. □

Proof of part (b).

$$\begin{aligned}(V^{\perp} + W^{\perp})^{\perp} &= (V^{\perp})^{\perp} \cap (W^{\perp})^{\perp} \quad \text{by part (a)} \\ &= V \cap W\end{aligned}$$

So $(V^{\perp} + W^{\perp}) = (V \cap W)^{\perp}$ □

Lemma 3.2.3. *Let V and W be vector spaces (not necessarily isotropic). Then*

$$V \subset W \iff W^{\perp} \subset V^{\perp}.$$

Proof. Suppose $V \subset W$ and let $x \in W^{\perp}$. Then $B(x, w) = 0$ for all $w \in W$. And $V \subset W$ so $B(x, v) = 0$ for all $v \in V$. Thus $x \in V^{\perp}$.

Now suppose $W^{\perp} \subset V^{\perp}$. Let $x \in V$. Then $B(x, y) = 0$ for all $y \in W^{\perp}$. This implies $x \in W$. □

3.2.2 Results for isotropic vector spaces

Lemma 3.2.4. *If V and M are maximal isotropic spaces, then $V^{\perp} \cap M^{\perp}$ is not isotropic.*

Proof. Let V and M be maximal isotropic subspaces ($\dim V = \dim M = n$). Suppose $V^{\perp} \cap M^{\perp}$ is isotropic. Then $V^{\perp} \cap M^{\perp} \subset V$ because otherwise the space $V + (V^{\perp} \cap M^{\perp}) \neq V$ is isotropic and would be larger than V. This contradicts maximality of V. Similary, $(V^{\perp} \cap M^{\perp}) \subset M$. So $(V^{\perp} \cap M^{\perp}) \subset (V \cap M)$. But an isotropic space is contained in its perp, so $V \subset V^{\perp}$ and $M \subset M^{\perp}$ so $(V \cap M) \subset (V^{\perp} \cap M^{\perp})$. Thus we have $(V \cap M) = (V^{\perp} \cap M^{\perp}) = (V + M)^{\perp}$.

$$\begin{aligned}\dim((V + M)^{\perp}) &= 2n + 1 - \dim(V + M) \\ &= 2n + 1 - \dim V - \dim M + \dim(V \cap M) \\ &= 2n + 1 - n - n + \dim(V \cap M) \\ &= 1 + \dim(V \cap M)\end{aligned}$$

So we have a contradiction and $V \cap M \neq (V + M)^{\perp}$. In particular, there is a vector e that is orthogonal to V and to M but not orthogonal to itself. So $e \in V^{\perp} \cap M^{\perp}$ which makes $V^{\perp} \cap M^{\perp}$ *not* isotropic. □

Corollary 3.2.1. *If A and B are isotropic spaces, then $A^{\perp} \cap B^{\perp}$ is not isotropic.*

Proof. Choose maximal isotropic spaces V and M such that $A \subset V$ and $B \subset M$. Then $A^{\perp} \cap B^{\perp}$ contains $V^{\perp} \cap M^{\perp}$ which is not isotropic by Lemma 3.2.4. □

Lemma 3.2.5. *For A and B isotropic spaces, we have*

$$A + B \textit{ is isotropic } \iff A \subset B^{\perp} \iff B \subset A^{\perp}$$

Proof. If $A+B$ is isotropic, then every vector in B is orthogonal to every vector in A, so $B \subset A^{\perp}$ and vice versa, $A \subset B^{\perp}$. If $A \subset B^{\perp}$ and A is isotropic, then every vector in A is orthogonal to every linear combination of vectors from A and B. And B isotropic with $A \subset B^{\perp}$ implies that every vector in B is orthogonal to every linear combination of vectors from A and B. So $A + B$ is isotropic. $\square$

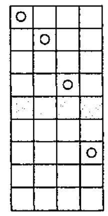


Figure 3.1: $k = n = 4$ and $0 < a_1 = 1 < a_2 = 2 < a_3 = 4 < a_4 = 7 < \infty$

3.3 Codimension bounds for Schubert conditions given by containment

In the main proof, we will need to consider Schubert conditions given by containment. Suppose we are given $1 \leq a_1 < a_2 < \cdots < a_n \leq 2n+1$ (with the convention that $a_0 = 0$ and $a_{n+1} = \infty$). In addition, $a_i + a_j \neq 2n+2$ for all i, j and $a_i \neq n+1$ for all i. We are also given integers j and R such that $a_j \leq R < a_{j+1}$.

We have a closed subvariety

$$T' \subset OFl(2n+1) \times OFl(2n+1)$$

which is defined by

$$T' = \{((V_i)_{i \leq n}, M_.) \mid V_i \subset M_{a_i}\}$$

T' can be constructed as a tower of quadrics over $OFl(2n+1)$ by inductively choosing $M_1, M_2, \ldots, M_n$ such that for $1 \leq k \leq n$

1. $M_{k-1} \subset M_k \subset M_{k-1}^{\perp}$ (with $\dim(M_k) = k$)

2. M_k is isotropic

3. $V_{i(k)} \subset M_k \subset V_{i(\underline{k})}^{\perp}$

where $i(a)$ is defined as in (4.2) and $\underline{k} = 2n+1-k$ (equation (2.2)). Then complete the flag $M_.$ by defining $M_{\underline{i}} = M_i^{\perp}$ for $0 \leq i \leq n$.

Consider a $(2n+1) \times k$ checker board. Stratify $OFl(1, \ldots, k, 2n+1) \times OFl(2n+1)$ by the numerical data $\dim(V_{i_2} \cap M_{i_1})$ for $1 \leq i_2 \leq k$. The strata correspond to checkerboards with k columns and $2n+1$ rows (think of removing the $2n+1-k$ columns without white checkers and renaming the remaining columns 1 through k), with k checkers, no two in the same row or column, no two in rows that sum to $2n+2$, and none in row $n+1$, such that $\dim(V_{i_2} \cap M_{i_1})$ is the number of checkers weakly northwest of (i_1, i_2). See figure 3.1 for an example. The condition that $a_j \leq R < a_{j+1}$ means R is a row between the j^{th} and $(j+1)^{st}$ checkers, possibly the same row as the j^{th} checker.

We can build the stratum as an open subset of a tower of quadric and projective bundles over $OFl(2n+1)$: quadric if the space that V_{i_2} sits in (i.e. M_{i_1}) is a perp space ($i_1 > n$), projective if M_{i_1} is isotropic ($i_1 \leq n$).

The dimension of the stratum is

$$\dim(OFl(2n+1)) + \sum_{\text{checker c at } (i_1,i_2)} (i_1 - \#\{\text{checkers weakly northwest of c}\} - \epsilon_{i_1}) \qquad (3.1)$$

where

$$\epsilon_{i_1} = \begin{cases} 1 & \text{if } i_1 > n \\ 0 & \text{if } i_1 \leq n \end{cases}$$

T' corresponds to configurations where there are at least i checkers in the first a_i rows, and the dense open stratum of T' corresponds to the configuration $\{(a_i, i) | 1 \leq i \leq n\}$. Note that this configuration is as much on the diagonal as possible. If the white checkers are less specialized, for example white checkers in $(1,2)$ and $(2,1)$ then there is no white checker in position $(1,1)$ which implies $\dim(V_1 \cap M_1) = 0$. But V_1 is supposed to be contained in M_1. In particular, there are j checkers in the first R rows.

Let B be a variety, $B \to OFl(2n+1)$ a morphism, and T'' the pullback of T' to B. Then we have the following lemma:

Lemma 3.3.1. *For any $\delta \leq k$, if P is an irreducible subvariety of T'' where* $\dim(V_\delta \cap M_R) = j + \ell_2$*, then* $\operatorname{codim}_{T''}(P) \geq \ell_2$*. Furthermore, if equality holds then one of the following is true:*

1. $\ell_2 = 0$
2. $\ell_2 = 1$, $R \geq n+2$, $a_j < R$, $a_{j+1} = R+1$, *and* $V_{j+1} \subset M_R$ *for all points of* P
3. $\ell_2 = 1$, $R < n$, $a_j < R$, $a_{j+1} = R+1$, *and* $V_{j+1} \subset M_R$ *for all points of* P
4. $\ell_2 = 1$, $R = n$, $a_j < R$, $a_{j+1} = n+2$, *and* $V_{j+1} \subset M_R$ *for all points of* P
5. $\ell_2 = 1$, $R = n+1$, $a_j < R$, $a_{j+1} = n+2$, *and* $V_{j+1} \subset M_R$ *for all points of* P

The proof of lemma 3.3.1 requires some preliminary lemmas connecting movement of checkers with codimension 1 Schubert conditions. In the type A_n setting, moving one white checker from row $i+1$ to row i is a codimension 1 condition on $\Lambda = \{M. \in Fl(n) \mid V_i \subset M_{a_i} \quad 1 \leq i \leq k\}$. This can be seen by looking at the forgetful map $\Lambda \longrightarrow \Lambda'$ which forgets M_i. The dimension of the fiber over a general point in Λ' is 1 because choosing M_i is equivalent to choosing a line in M_{i+1}/M_{i-1}. There is no condition from $V.$ because there is no white checker in row i which implies $V_{m(M_i)} = V_{m(M_{i-1})} \subset M_{i-1}$. When we move the white checker from row $i+1$ to row i, we require that $V_{m(M_{i+1})} \subset M_i$. Choosing M_i such that $M_{i-1} \subset M_i \subset M_{i+1}$ and $V_{m(M_{i+1})} \subset M_i$ is equivalent to choosing *the* line $(M_{i-1} + V_{m(M_{i+1})})/M_{i-1} \subset (M_{i+1}/M_{i-1})$. This is a codimension one condition on the one dimensional space.

In the B_n setting, we sometimes move two checkers or move one checker two rows. These moves correspond to codimension one conditions, however, the result is not immediate. We show here that such moves are indeed associated to codimension one conditions.

We define a variety $\Omega = \Omega_{a_1,\ldots,a_k}(V_1, \ldots, V_k)$. Suppose we are given V, a $(2n+1)$-dimensional vector space with a non-degenerate, symmetric bilinear form on it, an integer k with $1 \leq k \leq n$, and a sequence of increasing integers $0 = a_0 < a_1 < \cdots < a_k \leq 2n+1$ with

$$a_i + a_j \neq 2n+2 \text{ for any } 1 \leq i, j \leq k. \tag{3.2}$$

Finally, we are given an integer j with $1 \leq j \leq k$ such that $a_j - a_{j-1} \geq 2$. If there is an i such that $a_i = 2n+3-a_j$ then call this $a_{\bar{j}} = a_i$. Note that if $a_{\bar{j}}$ exists, then $a_{\bar{j}} - 1$ must be an empty row because row a_j has a white checker, so by equation (3.2), row $2n+2-a_j = a_{\bar{j}} - 1$ is empty. Consider the variety Y of pairs $(V., M.)$ of partial isotropic flags $V_1 \subset \cdots \subset V_k$ and full istropic flags $M.$ with $V_i \subset M_{a_i}$ for $1 \leq i \leq k$. This variety fibers over the variety of partial V-flags. We define the fiber over a flag $V. = (V_1 \subset \cdots \subset V_k)$

$$\begin{aligned} \Omega &= \Omega_{a_1,\ldots,a_k}(V_1, \ldots, V_k) \\ &= \{M. \in OFl(2n+1) \mid V_i \subset M_{a_i} \text{for } 1 \leq i \leq k\} \end{aligned}$$

By calculating the dimension of Ω for the white checker positions before and after the moves, we will show that the difference in dimension is one, and so the moves described correspond to codimension 1 conditions.

We now describe a formula for $\dim(\Omega)$ in terms of the a_i. For $1 \leq a \leq 2n+1$. We build Ω recursively as follows:

First choose M_n, maximal isotropic, such that $V_{i(n)} \subset M_n$. This is equivalent to the choice of an element of $OGr(n - i(n), V_{i(n)}^{\perp}/V_{i(n)})$ the dimension of which is $\frac{[n-i(n)][n-i(n)+1]}{2}$. Now, assuming $M_{a+1} \subset \cdots \subset M_n$ have been constructed, we next need to add M_a. It must satisfy the following three conditions.

1. $V_{i(a)} \subset M_a$

2. $M_a \subset M_{a+1}$

3. $V_{i(\underline{a})} \subset M_{\underline{a}} = M_a^{\perp}$ or equivalently $M_a \subset V_{i(\underline{a})}^{\perp}$

So we need to choose M_a satisfying $V_{i(a)} \subset M_a \subset (M_{a+1} \cap V_{i(\underline{a})}^{\perp})$. Note that isotropy is automatic because $M_a \subset M_n$ and M_n is isotropic.

There are two possibilities:

1. $i(\underline{a} - 1) = i(\underline{a})$. In this case, $M_{a+1} \subset V_{i(\underline{a})}^{\perp}$ by the previous step of the construction. The condition $i(\underline{a} - 1) = i(\underline{a})$ is equivalent to row $\underline{a}$ being empty.

2. $i(\underline{a} - 1) = i(\underline{a}) - 1$. In this case, for a sufficiently general choice of M_{a+1} in the previous step, the intersection has dimension a.

In Case 2, the current step adds zero to the dimension of Ω and in Case 1, the current step adds $\dim(\mathbb{P}(M_{a+1}/V_{i(a)})^*) = a - i(a)$ to the dimension of Ω. So we have a formula for $\dim(\Omega)$

$$\dim(\Omega) = \frac{[n - i(n)][n - i(n) + 1]}{2} + \sum_{a=1}^{n-1} \delta_a [a - i(a)] \tag{3.3}$$

where

$$\delta_a = \begin{cases} 1 & \text{if row } \underline{a} \text{ is empty} \\ 0 & \text{if there is a white checker in row } \underline{a}. \end{cases}$$

Let $\Omega = \Omega_{a_1,\ldots,a_k}(V_1, \ldots, V_k)$ be the variety corresponding to the white checker configuration before the move, and let $\Omega' = \Omega_{a'_1,\ldots,a'_k}(V_1, \ldots, V_k)$ be the variety corresponding to the new white checker configuration. Then

$$\dim(\Omega') = \frac{[n - i'(n)][n - i'(n) + 1]}{2} + \sum_{a=1}^{n-1} \delta'_a [a - i'(a)] \tag{3.4}$$

where

$$\delta'_a = \begin{cases} 1 & \text{if row } \underline{a} \text{ is empty in the new configuration} \\ 0 & \text{if there is a white checker in row } \underline{a} \text{ in the new configuration.} \end{cases}$$

Lemma 3.3.2. *A checker moved from row $n+2$ to row n corresponds to a codimension 1 condition.*

Call such a move a type M move.

Proof. For a move across the midline, we are given $a_1 < \cdots < a_k$ as above, and there is some j such that $a_j = n+2$ and $a_{j-1} < n$. Then $a'_i = a_i$ for $i \neq j$ and $a'_j = n$. We will show that $\dim \Omega - \dim \Omega' = 1$.

White checkers do not change positions in rows $1 \leq a \leq n-2$ nor in the corresponding rows $n+3 \leq \underline{a} \leq 2n+1$, so

$$\sum_{a=1}^{n-2} \delta_a [a - i(a)] = \sum_{a=1}^{n-2} \delta'_a [a - i'(a)].$$

This gives

$$\begin{aligned}\dim \Omega - \dim \Omega' = & [\frac{(n-i(n))^2 - (n-i(n))}{2} + \delta_{n-1}(n-1-i(n-1))] \\ & - [\frac{(n-i'(n))^2 - (n-i'(n))}{2} + \delta'_{n-1}(n-1-i'(n-1))].\end{aligned}$$

Now, $\delta_{n-1} = 0$ since row $2n+1-(n-1) = n+2 = a_j$ has a white checker in it. And $\delta'_{n-1} = 1$ since the white checker previously in row $n+2$ has moved to row n. In addition, $i'(n-1) = i(n-1) = i(n)$ and $i'(n) = i(n)+1$. This gives us

$$\begin{aligned}\dim \Omega - \dim \Omega' &= [\frac{(n-i(n))^2 - (n-i(n))}{2}] \\ &\quad - [\frac{(n-(i(n)+1))^2 - (n-(i(n)+1))}{2} + (n-1-i(n))] \\ &= 1.\end{aligned}$$

□

Lemma 3.3.3. *For $a \notin \{n, n+1, n+2\}$, the paired moves a to $a-1$ and $2n+3-a$ to $2n+1-a$ correspond to a codimension 1 condition.*

Call such a move a type P move.

Proof. Given $a_1 < \cdots < a_k$ as above and j such that $a_j \neq n+2$ and $a_{\bar{\jmath}} = \underline{a_j}$ as long as such an $a_i = a_{\bar{\jmath}}$ exists. We define $a'_i = a_i$ for $i \neq j, \bar{\jmath}$, $a'_j = a_j - 1$, and $a'_{\bar{\jmath}} = a_{\bar{\jmath}} - 1$ if $a_{\bar{\jmath}}$ exists. We will show that $\dim \Omega - \dim \Omega' = 1$.

For this situation, we are not moving a white checker into row n, so $i(n) = i'(n)$ which means $\frac{(n-i(n))^2-(n-i(n))}{2} = \frac{(n-i'(n))^2-(n-i'(n))}{2}$. The six rows of interest are rows $a_j - 2, a_j - 1, a_j$ and their corresponding rows $2n+3-(a_j-2), 2n+3-(a_j-1), 2n+3-a_j$. Other than these six rows, all other terms in the sum are the same for Ω and Ω'. We will consider four cases separately:

1. $a_j \leq n$ and row $2n+3-a_j$ is empty (i.e. $a_{\bar{j}}$ doesn't exist)

2. $a_j \leq n$ and row $2n+3-a_j$ has a white checker (i.e. $a_{\bar{j}}$ exists)

3. $a_j > n+2$ and row $2n+3-a_j$ is empty (i.e. $a_{\bar{j}}$ doesn't exist)

4. $a_j > n+2$ and row $2n+3-a_j$ has a white checker (i.e. $a_{\bar{j}}$ exists).

Case 1

$$\begin{aligned}\dim\Omega - \dim\Omega' = [&\delta_{a_j-2}(a_j - 2 - i(a_j - 2)) \\ &+ \delta_{a_j-1}(a_j - 1 - i(a_j - 1)) + \delta_{a_j}(a_j - i(a_j))] - [\delta'_{a_j-2}(a_j - 2 - i'(a_j - 2)) \\ &+ \delta'_{a_j-1}(a_j - 1 - i'(a_j - 1)) + \delta'_{a_j}(a_j - i'(a_j))]\end{aligned}$$

In Ω, row $2n+1-(a_j-2) = a_{\bar{j}}$ is empty by hypothesis, so $\delta_{a_j-2} = 1$. And row $2n+1-(a_j-1) = 2n+2-a_j$ is empty because it is the mirror of row a_j which has a white checker, so $\delta_{a_j-1} = 1$. In Ω', row $2n+1-(a_j-2) = a_{\bar{j}}$ is still empty so $\delta'_{a_j-2} = 1$ and row $2n+1-(a_j-1) = a_{\bar{j}}$ is empty because in Ω, row $2n+3-a_j$ was empty and so was $2n+2-a_j$. Nothing changes in row $2n+1-a_j$ so $\delta_{a_j} = \delta'_{a_j}$. Nothing has changed in rows through row a_j-2 so $i(a_j-2) = i'(a_j-2)$. A checker has moved into row $a_j - 1$ so $i(a_j-1) = i'(a_j-1)+1$. And finally, a checker moves from row a_j to row $a_j - 1$ so $i(a_j) = i'(a_j)$. This yields $\dim\Omega - \dim\Omega' = 1$.

Case 2

$$\begin{aligned}\dim\Omega - \dim\Omega' = [&\delta_{a_j-2}(a_j - 2 - i(a_j - 2)) \\ &+ \delta_{a_j-1}(a_j - 1 - i(a_j - 1)) + \delta_{a_j}(a_j - i(a_j))] - [\delta'_{a_j-2}(a_j - 2 - i'(a_j - 2)) \\ &+ \delta'_{a_j-1}(a_j - 1 - i'(a_j - 1)) + \delta'_{a_j}(a_j - i'(a_j))]\end{aligned}$$

Row $2n+1-(a_j-2) = a_{\bar{j}}$ has a white checker by hypothesis so $\delta_{a_j-2} = 0$. And as in case 1, $\delta_{a_j-1} = 1$. In Ω', row $2n+1-(a_j-2) = a_{\bar{j}}$ is empty because that checker has moved one row up to row $2n+1-(a_j-1)=a'_{\bar{j}}$, so $\delta'_{a_j-2}=1$ and $\delta'_{a_j-1}=0$. There is no change to row $2n+1-(a_j)$ so $\delta'_{a_j}=\delta_{a_j}$. And $i(a_j)=i'(a_j)$. Finally, there is no checker moving into row a_j-2 so $i'(a_j-2)=i(a_j-2)$. Again we have $\dim\Omega-\dim\Omega'=1$.

Case 3

Since $a_j>n+2$, the three terms in the sum that we are concerned about are $a=a_{\bar{j}}-2, a=a_{\bar{j}}-1$, and $a=a_{\bar{j}}$.

$$\begin{aligned}\dim\Omega-\dim\Omega' = {} & [\delta_{a_{\bar{j}}-2}(a_{\bar{j}}-2-i(a_{\bar{j}}-2)) \\ & +\delta_{a_{\bar{j}}-1}(a_{\bar{j}}-1-i(a_{\bar{j}}-1))+\delta_{a_{\bar{j}}}(a_{\bar{j}}-i(a_{\bar{j}}))]-[\delta'_{a_{\bar{j}}-2}(a_{\bar{j}}-2-i'(a_{\bar{j}}-2)) \\ & +\delta'_{a_{\bar{j}}-1}(a_{\bar{j}}-1-i'(a_{\bar{j}}-1))+\delta'_{a_{\bar{j}}}(a_{\bar{j}}-i'(a_{\bar{j}}))]\end{aligned}$$

Row $2n+1-(a_{\bar{j}}-2)=a_j$ is full, so $\delta_{a_{\bar{j}}-2}=0$. And row $2n+1-(a_{\bar{j}}-1)=a_j-1$ is empty, so $\delta_{a_{\bar{j}}-1}=1$. Moving the white checker from row a_j to row a_j-1 gives $\delta'_{a_{\bar{j}}-2}=1$ and $\delta'_{a_{\bar{j}}-1}=0$. Nothing changes in row $2n+1-a_{\bar{j}}=a_j-2$ so $\delta'_{a_{\bar{j}}}=\delta_{a_{\bar{j}}}$. And nothing changes in row $a_{\bar{j}}-2$ so $i'(a_{\bar{j}}-2)=i(a_{\bar{j}}-2)$. And finally by hypothesis, row $a_{\bar{j}}$ is empty in Ω so $i'(a_{\bar{j}})=i(a_{\bar{j}})$. So we have $\dim\Omega-\dim\Omega'=1$.

Case 4

$$\begin{aligned}\dim\Omega-\dim\Omega' = {} & [\delta_{a_{\bar{j}}-2}(a_{\bar{j}}-2-i(a_{\bar{j}}-2)) \\ & +\delta_{a_{\bar{j}}-1}(a_{\bar{j}}-1-i(a_{\bar{j}}-1))+\delta_{a_{\bar{j}}}(a_{\bar{j}}-i(a_{\bar{j}}))]-[\delta'_{a_{\bar{j}}-2}(a_{\bar{j}}-2-i'(a_{\bar{j}}-2)) \\ & +\delta'_{a_{\bar{j}}-1}(a_{\bar{j}}-1-i'(a_{\bar{j}}-1))+\delta'_{a_{\bar{j}}}(a_{\bar{j}}-i'(a_{\bar{j}}))]\end{aligned}$$

As in case 3, $\delta_{a_{\bar{j}}-2}=0$, $\delta_{a_{\bar{j}}-1}=1$, $\delta'_{a_{\bar{j}}-2}=1$, $\delta'_{a_{\bar{j}}-1}=0$, $\delta'_{a_{\bar{j}}}=\delta_{a_{\bar{j}}}$, $i'(a_{\bar{j}})=i(a_{\bar{j}})$, and $i'(a_{\bar{j}}-2)=i(a_{\bar{j}}-2)$. In Ω, row $a_{\bar{j}}-1$ is empty since it's the mirror of row a_j. So $i(a_{\bar{j}}-2)=i(a_{\bar{j}}-1)$. This yields $\dim\Omega-\dim\Omega'=1$. □

Lemma 3.3.4. *If there is no checker in row $2n+3-a$, then the move a to $a-1$ is a codimension 1 condition.*

Call such a move a type S move.

Proof. Moving one checker is just as in the A_n case, a codimension 1 condition. □

We now prove lemma 3.3.1 for $\delta=k\le n$ and $B\xrightarrow{\sim} OFl(1,\ldots,k,2n+1)$ (i.e. $T''=T'$).

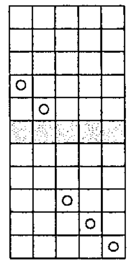


Figure 3.2: For the case $R \geq n+2$, where $n = 5$ and $\ell_2 = 3$. In this case, the set of rows weakly less than R that must be empty is $\{8, 7, 3\}$

Proof. A dense open set of P lies in some stratum where there are at least $j + \ell_2$ checkers in the first R rows. Our goal is to move ℓ_2 more white checkers into rows $1, \ldots, R$. There are already j white checkers in rows $1, \ldots, R$. We'd like to find the greatest lower bound for the codimension of the space $\{(V_., M_.)\}$ with the new constraints. This is based on the number of moves it will take to get the extra ℓ_2 white checkers into rows $1, \ldots, a_j$. We consider a *move* as one of the types M, P, and S described in lemmas 3.3.2, 3.3.3, and 3.3.4 respectively.

We begin with a checker configuration with exactly j checkers in the first R rows. We will calculate the minimal number of moves needed to get ℓ_2 more checkers into the first R rows. Since one move corresponds to a codimension one condition, counting moves is equivalent to calculating codimension.

Case $R \geq n+2$

The minimal number of moves will occur when the maximum ℓ_2 rows from the set of rows

$$\{R-i \mid 1 \leq R-i \leq R\} - \{\min(R-i, 2n+2-(R-i)) \mid R-i \text{ and } 2n+2-(R-i) \text{ are both in the previous set }\}$$

are empty. See figure 3.2 for an example. Each checker that is queued to rise will rise ℓ_2 rows (not including row $n+1$). Each row the checker moves up is a type M, P, or S move. There are ℓ_2 such checkers, yielding a total of ℓ_2^2 moves. So the minimal codimension of the new space is bounded above by ℓ_2^2. And $\ell_2^2 \geq \ell_2$.

Case $R \leq n$

The minimal number of moves will occur when there are at least ℓ_2 empty rows beginning with R and decreasing consecutively, and there are checkers in the minimum ℓ_2 available rows where

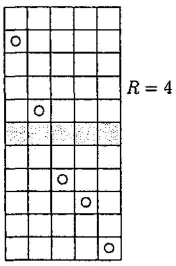


Figure 3.3: For the case $R \leq n$, where $n = 5$ and $\ell_2 = 2$. In this case, the set of rows greater than R that must have checkers is $\{5, 8\}$

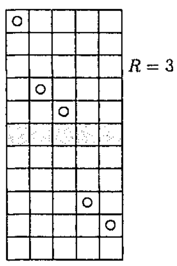


Figure 3.4: For the case $R \leq n$, where $n = 5$ and $\ell_2 = 2$. In this case, the set of rows greater than R that must have checkers is $\{4, 5\}$

available rows come from the set

$$\{R+i \mid R < R+i \leq 2n+1\} - \{\max(R+i, 2n+2-(R+i)) \mid R+i \text{ and } 2n+2-(R+i) \text{ are both in the previous set }\}$$

See figure 3.3 for an example. We have two subcases here.

1. $n \geq R + \ell_2$. In this case all the checkers queued to move into row R or above start above row $n+1$.

2. $n < R + \ell_2$. In this case, some of the white checkers queued to move into row R or above start in rows greater than $n+1$.

Subcase 1

See figure 3.4 for an example. Each checker that is queued to rise will rise ℓ_2 rows. Each row the checker moves up is a type P or S move. There are ℓ_2 such checkers, yielding a total of ℓ_2^2 moves. So the minimal codimension of the new space is bounded above by ℓ_2^2. And $\ell_2^2 \geq \ell_2$.

Subcase 2

See figure 3.5 for an example. In this case, the minimum ℓ_2 rows in our queued list are not all

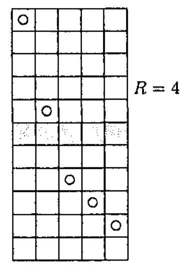


Figure 3.5: For the case $R \leq n$, where $n = 5$ and $\ell_2 = 3$. In this case, the set of rows greater than R that must have checkers is $\{5, 8, 9\}$

above row $n+1$. In particular, exactly $n - R$ of our queued checkers are above row $n+1$. We raise each of these $n - R$ checkers ℓ_2 rows. Each row the checker rises is a type P or S move. So we have a total of $(n - R)\ell_2$ moves. In fact, these moves are all type P and we now have $\ell_2 - (n - R)$ checkers in rows $n+2, n+3, \ldots, 1+\ell_2+R$. If there is a type S move, the there is an empty row somewhere in rows $n+2, n+3, \ldots, 1+\ell_2+R$ and we would have to add additional moves. So we may assume all of these moves are of type P.

For the checker now in row $n+2$, we perform a type M move to bring it to row n. Then we perform $\ell_2 - 1$ type P moves by raising this checker $\ell_2 - 1$ rows. Note that the checkers in rows $n+3, \ldots, R+1+\ell_2$ are now in rows $n+2, \ldots, R+1+\ell_2-1$. In general, for the i^{th} checker that is queued below row $n+2$ (after the initial $(n-R)\ell_2$ moves), we perform one type M move and $\ell_2 - i$ type P moves, for a total of $\ell_2 - i + 1$ moves. So the total moves needed to add ℓ_2 checkers into row R or above is at least

$$(n - R)\ell_2 + \sum_{i=1}^{\ell_2-(n-R)} (\ell_2 - i + 1) = \frac{1}{2}\ell_2^2 + \frac{1}{2}\ell_2 + (n - R)[-\frac{1}{2} + \ell_2 - \frac{1}{2}(n - R)]$$

Here, both $n - R$ and $-\frac{1}{2} + \ell_2 - \frac{1}{2}(n - R)$ are non-negative. For the minimum total moves to equal ℓ_2 one of the following must occur.

1. $\ell_2 = 0$ (which forces $n = R$).

2. $\ell_2 = 1$ with $n = R$

3. $\ell_2 = 1$ with $-\frac{1}{2} + \ell_2 - \frac{1}{2}(n - R) = 0$. This implies $R = n - 1$, but $n < R + \ell_2$ which implies $n < n$, a contradiction.

So in this case, if we have equality, either $\ell_2 = 0$ or $\ell_2 = 1$ with $n = R$.

In these calculations, we've always kept checkers in increasing rows across increasing columns. Suppose we move to a stratum that can be described by a checker configuration that has some checkers in non-increasing rows (as columns increase). We are only interested in minimal numbers of moves and so in the case of equality ($\ell_2 =$ minimal number of moves) we consider the cases where $\ell_2 = 1$.Then we have the $(j+1)^{st}$ checker in row $R+1$ that we'd like to move into row R. If we move any of checkers numbered $j+2, j+3, \ldots$ into row R, we must move them more that one row (in order to bypass checker $j+1$). Thus we no longer have a minimal number of moves. So the only way for ℓ_2 to be the number of moves is to move checker $j+1$ into row R. This forces $V_{j+1} \subset M_R$ for all points in P.

Finally, for the case $R = n+1$, we know $V_\delta \cap M_{n+1} = V_\delta \cap M_n$ since V_δ is isotropic. So if $R = n+1$, we still need to move checkers into row n, so $R = n$ is equivalent to $R = n+1$. $\square$

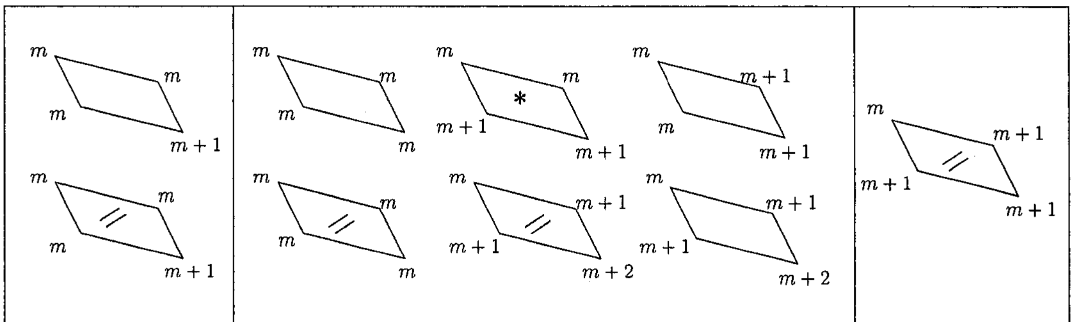


Figure 3.6: Possible labeled quadrilaterals, where $\mathbf{j}$ is labeled with $\dim(V_j \cap M)$ for some fixed vector space M. (Quadrilateral $*$ arises in lemma 3.4.1)

3.4 Content of quadrilaterals

Lemma 3.4.1 discusses content of a quadrilateral from a quilt $Q_\circ$. Suppose we are given a vector space $M \subset \mathbb{C}^{2n+1}$ and an element $(V_m)_{\mathbf{m}\in Q_\circ}$ of $OBS(Q_\circ)$. We label each element $\mathbf{m}$ of the quilt with $\text{label}(\mathbf{m}) = \dim(V_m \cap M)$. For each quadrilateral in $Q_\circ$, define the *content* of the quadrilateral as

$$\text{label}(\mathbf{m}_{NE}) + \text{label}(\mathbf{m}_{SW}) - \text{label}(\mathbf{m}_{NW}) - \text{label}(\mathbf{m}_{SE}) \tag{3.5}$$

where $\mathbf{m}_{NE}$ is the northeast element of the quadrilateral, $\mathbf{m}_{SW}$ is the southwest element of the quadrilateral, $\mathbf{m}_{NW}$ is the northwest element of the quadrilateral, and $\mathbf{m}_{SE}$ is the southeast element of the quadrilateral. The quilts we are working with have a maximal space V that is isotropic, so all subspaces associated to the quilt $Q_\circ$ are isotropic without extra conditions. So Lemma 5.5 in [22] can be used in the B_n case directly. We restate the lemma here (see also Figure 3.6):

Lemma 3.4.1. *Suppose we are given a locally closed subvariety*

$$U \subset Fl(1,\ldots,k,n) \times G(R,n) = ((V_j), M_R)$$

where the rank data $(V_j \cap M_R)_{1\le j\le k}$ *is constant, and* $(V_j)_{1\le j\le k}$ *corresponds to the northwest border of some given* $Q_\circ$. *Define* P *via the pullback diagram*

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d] & BS(Q_\circ)_S \times G(R,n) \arrow[d] \\ U \arrow[r, hook] & Fl(1,\ldots,k,n) \times G(R,n) \end{tikzcd}$$

where $BS(Q_\circ)_S$ is a given open stratum of $BS(Q_\circ)$ (and elements of S are marked with "$=$"). Let $((V_m)_{\mathbf{m}\in Q_\circ}, M_R)$ be a general point of P. Label $\mathbf{m}$ with $\dim(V_m \cap M_R)$.

(a) Then no quadrilaterals of type $*$ in Figure 3.6 appear.

(b) Assume furthermore that no negative-content quadrilaterals appear, and all quadrilaterals marked "$=$" have positive content.

(i) If the northern two vertices of a quadrilateral are labeled m, then the southern two vertices are also labeled m, and the quadrilateral is not marked "$=$".

(ii) If the western two vertices of a quadrilateral are labeled m, then the eastern two edges are labeled the same (both m or $m+1$), and the quadrilateral is not marked "$=$".

Proof. See proof of lemma 5.5 in [22]. $\square$

We include an additional observation here involving content of quadrilaterals.

Lemma 3.4.2. *If we have a content 1 quadrilateral with associated vector spaces $V_{NW}, V_{NE}, V_{SW}, V_{SE}$ labeled $m, m+1, m+1, m+1$ respectively where the label is given by $\dim(V_\alpha \cap M_r)$. Then $V_{NE} = V_{SW}$.*

Proof. $\dim(V_{NE} \cap M_r) = \dim(V_{SW} \cap M_r) = \dim(V_{SE} \cap M_r) = m+1$ means that $V_{NE} \cap M_r = V_{SW} \cap M_r$ and there is a line L with $L \subset V_{NE} \cap M_r = V_{SW} \cap M_r$ such that $V_{NE} \cap M_r = (V_{NW} \cap M_r) \oplus L$ and $L \not\subset V_{NW} \cap M_r$. Now, $L \subset M_r$ since $L \subset V_{NE} \cap M_r$, so it must be that $L \not\subset V_{NW}$. Since $\dim V_{NE} = \dim V_{NW} + 1$ we have $V_{NE} = V_{NW} \oplus L$. Similarly $V_{SW} = V_{NW} \oplus L$. So $V_{NE} = V_{SW}$. $\square$

Chapter 4

PARTIAL PROOF OF THE TYPE B_N GEOMETRIC LITTLEWOOD-RICHARDSON RULE

4.1 Strategy of proof

The overall strategy of the proof of conjecture 3 is the same as developed by Vakil in [22]: Instead of considering the divisor D on the closure of $X_{\circ\bullet}$ in $OGr(n, 2n+1) \times (X_\bullet \cup X_{\bullet_{next}})$, we consider the corresponding divisor D_Q on the closure of $OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$. See diagram (4.1). The map π is the projection from a point $(V, M_., F_.) \in \mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ to $(V, M_., F_.) \in \mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ that drops all subspaces associated to the quilt except $V_{\max}$.

$$\begin{tikzcd}
Cl_{OBS(Q_\circ)\times X_\bullet} X_{\circ\bullet} \arrow[r, hook, "\text{open}"] \arrow[d] & \mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet} \arrow[d, "\pi"'] & D_Q \arrow[l, hook', "\text{closed}"'] \arrow[d] \\
Cl_{OGr(n,2n+1)\times X_\bullet} X_{\circ\bullet} \arrow[r, hook, "\text{open}"] \arrow[d] & \mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet} \arrow[d] & D \arrow[l, hook', "\text{closed}"'] \arrow[d] \\
X_\bullet \arrow[r, hook, "\text{open}"] & X_\bullet \cup X_{\bullet_{next}} & X_{\bullet_{next}} \arrow[l, hook', "\text{closed}"']
\end{tikzcd} \tag{4.1}$$

In this chapter, we prove certain cases of the type B_n geometric Littlewood-Richardson rule (conjecture 3). In each case, we prove the following.

1. In section 4.2 we show the result holds in the trivial cases. For s_i moves, this occurs when there is no white checker in either of the descending black checker rows (rows d_E and d_W). And for s_0 moves, this occurs when there is no white checker in row n. Following this section, we will assume there is a white checker in row n for s_0 moves and a white checker either in row d_E or d_W or both for s_i moves.

2. We describe $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ as the intersection of two spaces, $W_\circ$ and $W_{\bullet\bullet_{next}}$. This is theorem 4.4.1.

3. We identify the irreducible components $\{D_S\}$ of D_Q (theorem 4.5.1).

4. We show all but one or two of the D_S are contracted by π (section 4.6). The statement of the theorems in this section vary depending on the case.

5. We then prove theorem 4.7.1 which states that in the remaining irreducible components, the multiplicity of D_Q is one.

6. Finally, in theorem 4.8.1 we show the D_S map birationally to $X_{\circ_{stay}\bullet_{next}}$ or $X_{\circ_{swap}\bullet_{next}}$, giving us the expected answers which occur with multiplicity one.

4.2 Proof of the rule in trivial cases

4.2.1 Trivial case for s_i moves

The trivial case for s_i moves, $i \neq 0$ occurs when there is no white checker in row d_E and no white checker in row d_W (recall definition 11).

Theorem 4.2.1. *For an s_i move, if there is no white checker in row d_E and no white checker in row d_W then $D = X_{\circ_{stay}\bullet_{next}}$.*

Proof.

Case 1: $d_E < n$

Let $X'_{\circ\bullet}$ be the projection of

$$X_{\circ\bullet} \to OGr(n, 2n+1) \times OFl(1, \cdots, \hat{d_E}, \cdots, \hat{d_W}, \cdots, 2n+1) \times OFl(2n+1)$$

by forgetting M_{d_E} and M_{d_W}. Note that $M_{d_W} = M_{d_E}^{\perp}$ since $2n+1-d_E = d_W$. In this case we assume that $d_E < n$.

To recover $X_{\circ\bullet}$, we can choose M_{d_E} such that $M_{d_E-1} \subset M_{d_E} \subset M_{d_E+1}$. Since $M_{d_E+1} \subset M_n$, M_{d_E} will automatically be isotropic because M_n is. The full $\mathbb{P}^1$ of choices for M_{d_E} gives $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}}\mathrm{X}_{\circ\bullet}$, the $\mathbb{P}^1$-bundle over $X'_{\circ\bullet}$ corresponding to choosing M_{d_E} as above. The extra point in each fiber is the point at infinity, i.e. the choice of M_{d_E} with the property that $(M_{d_E} \cap F_c^{\perp}) \subset F_{c+1}^{\perp}$, so D is the section of $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}}\mathrm{X}_{\circ\bullet}$ given by $\{M_{d_E} \mid (M_{d_E} \cap F_c^{\perp}) \subset F_{c+1}^{\perp}\}$. Two loose ends remain to be checked:

1. The choice of M_{d_E} is completely independent of $V \in OGr(n, 2n+1)$ when V is described by a white checker configuration with no white checkers in rows d_E nor d_W. In other words, V imposes no conditions on M_{d_E}, so any choice of M_{d_E} such that $M_{d_E-1} \subset M_{d_E} \subset M_{d_E+1}$ will yield a point $(V, M_., F_.) \in \mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}}\mathrm{X}_{\circ\bullet}$.

2. $D = \overline{X}_{\circ_{stay}\bullet_{next}}$

Choosing M_{d_E} is independent of V: Let $0 < a_1 < a_2 < \cdots < a_n \leq 2n+1$ be the n rows with white checkers given by the $\circ$-configuration. Define

$$i(a) = \max\{i \mid a_i \leq a\}. \tag{4.2}$$

For $(V, M., F.) \in Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$, we have the properties $\dim(V \cap M_a) \geq i(a)$ for $1 \leq a \leq 2n+1$ and $M.$ and $F.$ meet in either the $\bullet$-way or the $\bullet_{next}$-way. Now the fiber over the projection of $(V, M., F.)$ is the set of all M_{d_E} such that $\dim(V \cap M_{d_E}) \geq i(d_E)$, $\dim(V \cap M_{d_W}) \geq i(d_W)$, $M_{d_E - 1} \subset M_{d_E} \subset M_{d_E+1}$, and $(M_{d_E-1} \cap F_c^\perp) \subset F_{c+1}^\perp$.

Now, $\dim(V \cap M_{d_E-1}) \geq i(d_E - 1) = i(d_E)$ since there is no white checker in row d_E. And $\dim(V \cap M_{d_W-1}) \geq i(d_W - 1) = i(d_W)$ since there is no white checker in row d_W. So $M_{d_E-1} \subset M_{d_E}$ implies that $\dim(V \cap M_{d_E}) \geq i(d_E)$ and $M_{d_W-1} \subset M_{d_W}$ implies that $\dim(V \cap M_{d_W}) \geq i(d_W)$. So no knowledge of V is needed to choose M_{d_E}.

The divisor D: We have a $\mathbb{P}^1$ of choices for M_{d_E} where $M_{d_E-1} \subset M_{d_E} \subset M_{d_E+1}$. The single choice of M_{d_E} such that $(M_{d_E} \cap F_c^\perp) \subset F_{c+1}^\perp$ gives the point over $X_{\bullet_{next}}$. So $\overline{X}_{\circ_{stay}\bullet_{next}}$ is the section given by the divisor

$$D = \{M_{d_E} \mid (M_{d_E} \cap F_c^\perp) \subset F_{c+1}^\perp\}$$

See diagram (4.3) where $s(X'_{\circ\bullet}) \cong D$.

$$\begin{tikzcd} X_{\circ\bullet} \arrow[r, hook] \arrow[d, "p_{d_E}"'] & Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet} \\ X'_{\circ\bullet} \arrow[ur, "s"'] & \end{tikzcd} \tag{4.3}$$

Case 2: $d_E \geq n+2$

Again, let $X'_{\circ\bullet}$ be the projection of

$$X_{\circ\bullet} \to OGr(n, 2n+1) \times OFl(1, \ldots, \hat{d_W}, \ldots, \hat{d_E}, \ldots, 2n+1) \times OFl(2n+1)$$

by forgetting M_{d_W} and M_{d_E}. Note that $M_{d_E} = M_{d_W}^\perp$ since $2n + 1 - d_W = d_E$. In this case, we assume that $d_W < n$.

To recover $X_{\circ\bullet}$, we can choose M_{d_W} such that $M_{d_W-1} \subset M_{d_W} \subset M_{d_W+1}$. Since $M_{d_W+1} \subset M_n$, M_{d_W} will automatically be isotropic because it sits inside an isotropic space. If we look at the full $\mathbb{P}^1$ of choices for M_{d_W} then we get more than just $X_{\circ\bullet}$, we get $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$. And we say $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ is the $\mathbb{P}^1$-bundle over $X_{\circ\bullet}$ corresponding to choosing M_{d_W} as above. The extra point in each fiber is the point at infinity. It is the choice of M_{d_W} with the property that $(M_{d_W}^\perp \cap F_c^\perp) = (M_{d_E} \cap F_c^\perp) \subset F_{c+1}^\perp$, so D is the section of $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ given by $\{M_{d_W} \mid (M_{d_W}^\perp \cap F_c^\perp) \subset F_{c+1}^\perp\}$. The rest of this proof is almost exactly the same as the case $d_E < n$, replacing d_E with d_W. $\square$

4.2.2 Trivial case for s_0 moves

The trivial case for s_0 moves occurs when there is no white checker in row n.

Theorem 4.2.2. *For an s_0 move, if there is no white checker in row n then $D = X_{\circ_{stay}\bullet_{next}}$.*

Proof. Let $X'_{\circ\bullet}$ be the image of the projection

$$p_n : X_{\circ\bullet} \to OGr(n, 2n+1) \times OFl(\hat{n}) \times OFl(2n+1)$$

by forgetting M_n and $M_n^{\perp} = M_{n+1}$. For a general point $p_n(t) \in p_n(X_{\circ\bullet})$, the fiber $p_n^{-1}(t)$ is isomorphic to an open subset of $Y = O\mathbb{P}(M_{n+2}/M_{n-1}) \subset \mathbb{P}(M_{n+2}/M_{n-1})$. The closure of this set in $OGr(n, 2n+1) \times (X_\bullet \cup X_{\bullet_{next}})$ is all of Y. Since there are no white checkers in row n or $n+1$, there are no restrictions coming from $V \in OGr(n, 2n+1)$ on the choice of M_n.

Consider the section s of p_n, defined as follows: for the point

$$p_n(t) = (V, M_1 \subset \cdots \subset M_{n-1}, F_.)$$

we need M_n such that

$$M_{n-1} \subset M_n \subset M_{n-1} + (M_{n+2} \cap F_{c+1}^{\perp})$$

where $\dim(M_n) = n$ and M_n is isotropic.

Let $L = M_{n+2} \cap F_{c+1}$. By the $\bullet$-configuration, L has dimension 1. Define $M_n = M_{n-1} + L$. This choice of M_n gives us a point in $OGr(n, 2n+1) \times X_{\bullet_{next}}$:

$$\begin{aligned}(M_., F_.) \in X_{\bullet_{next}} &\iff \dim(M_n \cap F_c^{\perp}) = \dim(M_n \cap F_{c+1}^{\perp}) \\ &\iff \dim((M_{n-1} + L) \cap F_c^{\perp}) = \dim((M_{n-1} + L) \cap F_{c+1}^{\perp})\end{aligned}$$

which are equal since $L \subset F_{c+1} \subset F_{c+1}^{\perp} \subset F_c^{\perp}$ and $M_{n-1} \cap F_c^{\perp} = M_{n-1} \cap F_{c+1}^{\perp}$ in the $\bullet$-configuration. We now show that this choice for for M_n is the unique choice such that $(M_., F_.) \in X_{\bullet_{next}}$ instead of in $X_\bullet$.

By the black checker configuration, we have

$$M_{n+1} \cap F_c^{\perp} = M_{n+1} \cap F_{c+1}^{\perp} \iff (M_., F_.) \in X_{\bullet_{next}}$$

Now, $M_{n+1} \cap F_c^{\perp} = (M_n + F_c)^{\perp}$ and $M_{n+1} \cap F_{c+1}^{\perp} = (M_n \cap F_{c+1})^{\perp}$. So

$$\begin{aligned}(M_., F_.) \in X_{\bullet_{next}} &\iff M_n + F_c = M_n + F_{c+1} \\ &\iff \dim(M_n \cap F_c) = \dim(M_n \cap F_{c+1}) - 1 \\ &\iff \text{there is a line } \mathcal{L} \subset F_{c+1} \text{ such that } M_n = M_{n-1} + \mathcal{L}\end{aligned}$$

And $M_n \subset M_{n+2}$ means $\mathcal{L} \subset M_{n+2}$. So $\mathcal{L} \subset M_{n+2} \cap F_{c+1}$. Now, $\dim(M_{n+2} \cap F_{c+1}) = 1$, so $\mathcal{L}$ is unique and $\mathcal{L} = L$. We've shown there is a unique divisor D in $Cl_{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})} X_{\circ\bullet}$ that satisfies the conditions for $X_{\bullet_{next}}$.

We now show that the divisor D has multiplicity 1. We give a test family $\mathcal{F}$ through a general point $t \in (V, M_., F_.)$ of $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ meeting the divisor $D = X_{\circ_{stay}\bullet_{next}}$ with multiplicity 1. For our general point of $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$, we know $(M_., F_.) \in X_\bullet$. We choose a basis for $F_.$ and $M_.$:

- Let $F_.$ have the standard basis with $F_j = \langle e_1, \ldots, e_j \rangle$.
- Let $M_.$ have the basis that depends on the $\bullet$-configuration. In particular,

$$M_{n-1} = \langle e_{2n+1}, e_{2n}, \ldots, e_{2n+2-c}; e_{c+2}, e_{c+3}, \ldots, e_n \rangle$$

  and

$$M_{n+2} = M_{n-1} + \langle e_{\underline{c}}, e_{n+1}, e_{c+1} \rangle.$$

Build the one-dimensional test family $\mathcal{F} = \{(V', M'_., F'_.)\}$ as follows:

- Let $V' = V$
- Let $F'_. = F_.$
- For $1 \leq i \leq n-1$, let $M'_i = M_i$ and $(M'_i)^\perp = M_i^\perp$
- This leaves M'_n and $M'_{n+1} = (M'_n)^\perp$. Define

$$M'_n = \langle M_{n-1}, -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1} \rangle$$

  for $[s,t] \in \mathbb{P}'$. M'_n is isotropic because $\langle M_{n-1}, -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1} \rangle \subset M_{n+2} = M_{n-1}^\perp$ and $\langle -\frac{1}{2}s^2 e_{\underline{c}} + st e_{n+1} + t^2 e_{c+1} \rangle$ is itself isotropic.

We define the family $\mathcal{F}$ to be the the open subset of $\{(V', M'_., F'_.)\}$ described above where $t \neq 0$. When $s \neq 0$ then $(M'_., F'_.) \in X_\bullet$ so $\mathcal{F} \not\subset D$. And when $[s,t] = [0,1]$ then $(M'_., F'_.) \in X_{\bullet_{next}}$. So $\mathcal{F}$ meets D.

The divisor D on $\mathcal{F}$ is given by

$$\begin{aligned} M'_n \cap F_c^\perp \subset F_{c+1}^\perp &\iff \dim(M'_n \cap F_{c+1}) = 1 \\ &\iff \dim((M_{n-1} + \mathcal{L}) \cap F_{c+1}) = 1 \end{aligned}$$

Now, in both $X_\bullet$ and $X_{\bullet_{next}}$ we have that $\dim(M_{n-1} \cap F_{c+1}) = 0$, so $\dim((M_{n-1} + \mathcal{L}) \cap F_{c+1}) = 1 \iff \mathcal{L} \subset F_{c+1}$. This is equivalent to

$$\langle -\frac{1}{2}s^2 e_{\underline{c}} + s e_{n+1} + e_{c+1} \rangle \subset F_{c+1},$$

which is true if and only if $-\frac{1}{2}s^2 = 0$ and $s = 0$, a multiplicity one condition. So $\mathcal{F}$ meets D with multiplicity one and thus D has multiplicity one.

$\square$

4.3 A description of $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}}\mathrm{X}_{\circ\bullet}$

Similar to [22], we describe a closed subscheme of $OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$ and show it is $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet\cup X_{\bullet_{next}})}}\mathrm{X}_{\circ\bullet}$ (this is theorem 4.4.1). The subscheme will be constructed as the intersection of two subvarieties of an open subset of a tower of projective and quadric bundles over $OBS(Q_\circ)$.

A note on abusive notation: We say $X_{\circ\bullet} \subset OBS(Q_\circ) \times X_\bullet$ when in fact, $X_{\circ\bullet}$ is a subset of $OGr(n, 2n+1) \times X_\bullet$. There is however, a natural injection

$$X_{\circ\bullet} \hookrightarrow OBS(Q_\circ) \times X_\bullet$$

which takes $(V, M., F.) \mapsto ((V_m)_{\mathbf{m}\in Q_\circ}, M., F.)$ such that for $\mathbf{m} \in Q_\circ$ in position (i,j) on the checker board, $V_m = V \cap M_i \cap F_j$. So without further note, we say $X_{\circ\bullet} \subset OBS(Q_\circ) \times X_\bullet$ and leave the injection implicit.

4.3.1 How to build T

Definition 14. Let $\mathbf{m(M_i)}$ $(1 \le i \le 2n+1)$ be the maximum element $\mathbf{m}$ of $Q_\circ$ in rows up through i. Define $\mathbf{m(F_j)}$ similarly to be the maximum element $\mathbf{m} \in Q_\circ$ in columns up through j. In particular, we define $\mathbf{a} = \mathbf{m(F_{\underline{c+1}})}$ (c is defined in definition 12), an element we will refer to often.

We've already dealt with the trivial cases so we will assume for the remainder of the discussion that there is a white checker in row n (for the s_0 moves) or there is a white checker in row d_E or d_W or both (for the s_i moves).

Remark: We assume also that there is no white checker in column $c+1$. A white checker in columns $c+1$ requires more thought and will be determined later.

Consider a subspace

$$T \subset OBS(Q_\circ) \times OFl(2n+1) \times OFl(1, \cdots, c; \underline{c}, \cdots, 2n+1).$$

We describe how to build T and discuss some of its properties. We will then define spaces $Q, W_\circ$, and $W_{\bullet\bullet_{next}}$ which are fibered over T.

T is built like this:

Start with the base space $OBS(Q_\circ)$. For a point $(V_\alpha)_{\alpha\in Q_\circ} \in OBS(Q_\circ)$, build $M.$ in the following way "from outside to inside." Let $M_0 = \langle 0\rangle$, then for $1 \le i \le n$, choose M_i such that

1. $M_{i-1} \subset M_i \subset M_{i-1}^{\perp}$

2. M_i is isotropic

3. $V_{m(M_i)} \subset M_i \subset V_{m(M_i)}^{\perp}$

Complete the isotropic flag $M.$ by defining for $0 \leq i \leq n$, $M_{2n+1-i} = M_i^{\perp}$.

For a point $((V., M.)$, build the partial isotropic flag $F_{\leq c}$ in a similar way to $M.$. Let $F_0 = \langle 0 \rangle$, then for $1 \leq j \leq c$, choose F_j such that

1. $F_{j-1} \subset F_j \subset F_{j-1}^{\perp}$

2. F_j is isotropic

3. $V_{m(F_j)} \subset F_j \subset V_{m(F_j)}^{\perp}$

4. F_j is transverse to the flag $M.$

Then for $0 \leq j \leq c$ define $F_{2n+1-j} = F_j^{\perp}$. This completes the space T.

Theorem 4.3.1. *At each midsort (conjecture 1) step in the degeneration, the space T is reduced and irreducible.*

Proof. We build T on $OBS(Q_\circ)$ by choosing $M_1, M_2, \ldots, M_n$ and then $F_1, \ldots, F_c$. If there is no white checker in row $i+1$ then choosing M_{i+1} is equivalent to choosing an isotropic line in $(M_i^{\perp} \cap V_{m(M_{i+1})}^{\perp})/M_i$. If there is a white checker in row $i+1$ then there is exactly one choice for M_{i+1}, namely $M_{i+1} = M_i + V_{m(M_{i+1})}$. We choose the F_j's in a similar way with the additional open condition that F_j is transverse to the $M.$ flag.

We'd like to show that T is reduced and irreducible. Together these amount to showing that at each step where we add M_{i+1} when there is no white checker in row $i+1$, that the rank of the symmetric bilinear form, $rank(B)$, on $(M_i^{\perp} \cap V_{m(M_{i+1})}^{\perp})/M_i$ is greater than or equal to 3.

Let $W = V_{m(M_{i+1})}^{\perp}$, $\dim(W) = k$, $rank(B|_W) = r$ where r is odd. Also let $V = W^{\perp} = V_{m(M_{i+1})}$ and $M = M_i$. Note that $V_{m(M_{i+1})} \subset V_{m(M_i)} \subset M_i^{\perp}$ so $M = M_i \subset V_{m(M_{i+1})}^{\perp} = W$. Our claim is now reworded: $rank(B)$ on $(M^{\perp} \cap W)/M$ is greater than or equal to 3.

We do a change of basis so that our form B is nicer looking. See Figure 4.1. r is odd so let $r = 2q+1$. So the center 1 of the $r \times r$ block (call this block R) of the matrix is in position

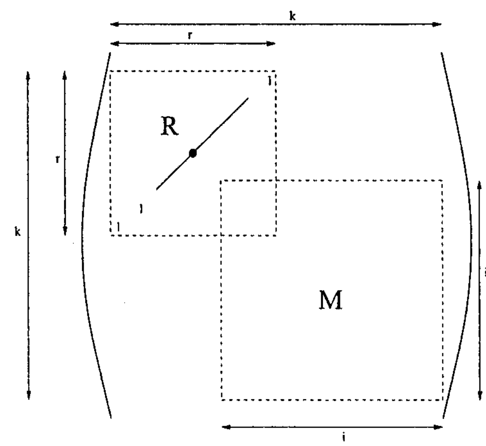


Figure 4.1: The form for the change of basis matrix.

$(q+1, q+1)$. M is isotropic so the block M cannot meet the antidiagonal of ones in block R. Thus we must have $i+q+1 \leq k$. This implies that $i < k-q$.

We have two cases to consider depending on if blocks R and M overlap or not.

Case 1

They do not overlap, i.e. $i \leq k-r$. Then $M_i^{\perp} = W$ and $\text{rank}((M_i^{\perp} \cap W)/M_i) = r$. Since r is odd, either $r=1$ or $r \geq 3$. If $r=1$ then $V_{m(M_{\underline{i+1}})}$ is maximal isotropic. This is because $W = V^{\perp}_{m(M_{\underline{i+1}})}$ is the perp of an isotropic space and if $rank(B|_W) = 1$, then we've only added the middle vector so $V_{m(M_{\underline{i+1}})}$ must have been dimension n. This means there are n white checkers in rows $1, \ldots, \underline{i+1}$. If there are n white checkers in rows $1, \ldots, \underline{i+1}$ then we must be in the maximal case. And if we are in the maximal case and we've assumed no white checker in row $i+1$, then there must be a white checker in row $2n+2-(i+1) = 2n+1-(i+1)+1 = \underline{i+1}+1$. So there cannot be n white checkers in rows $1, \ldots, \underline{i+1}$. Thus $r \neq 1$, which implies $r \geq 3$.

Case 2

Blocks R and M overlap, i.e. $i > k-r$. See Figure 4.2. M and $M^{\perp}$ are spanned by the basis vectors

$$M = \langle e_k, e_{k-1}, \ldots, e_{k-i+1} \rangle$$
$$M^{\perp} = \langle e_k, \ldots, e_{k-i+1}, e_{k-i}, \ldots, e_{r+1-(k-i)} \rangle$$

so in the quotient space we have

$$(M^{\perp} \cap W)/M = \langle \overline{e}_{r+1-(k-i)}, \ldots, \overline{e}_{k-i} \rangle$$

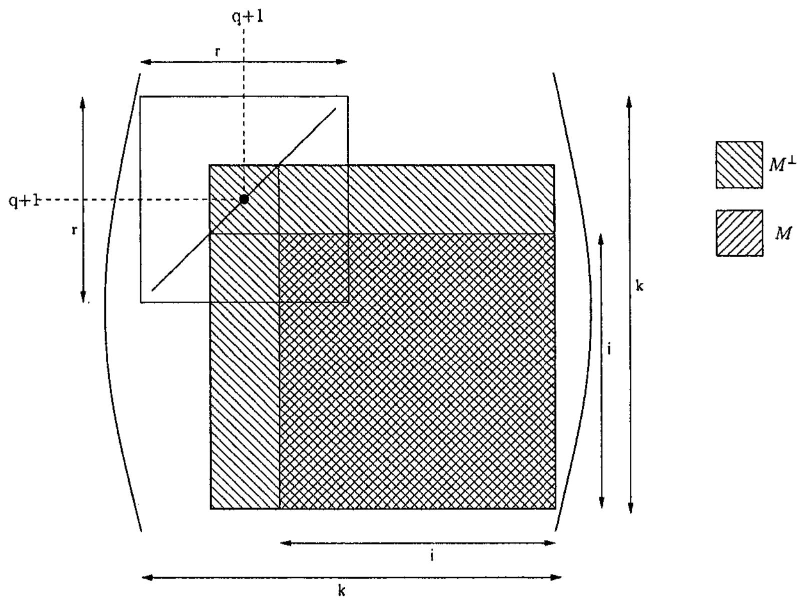


Figure 4.2: This is the form of the change of basis matrix if M and R blocks overlap, i.e. $i > k - r$

and

$$\begin{aligned} rank(\overline{B}) &= (k-i) - (r+1-(k-i)) + 1 \\ &= 2(k-i) - r \end{aligned}$$

We know $r = 2q+1$ and by hypothesis we have both $i > k-r$ and $i < k-q$. This implies $2(k-i) - r \geq 0$. And $2(k-i) - r$ is odd because r is odd and $2(k-i)$ is even. We again consider the two possibilities $2(k-i) - r = 1$ and $2(k-i) - r \geq 3$. Suppose $2(k-i) - r = 1$. Then we have

$$\begin{aligned} 2(k-i) - r &= 1 \\ 2(k-i) &= r + 1 \\ 2(k-i) &= (2q+1) + 1 \\ k - i &= q + 1 \end{aligned}$$

This means the upper left corner of the M block is at position $(q+2, q+2)$.

Recall that $i(m)$ is the number of white checkers in rows $1, \ldots, m$ of the checker board. Then

$$\begin{aligned} k &= \dim W \\ &= 2n + 1 - \dim(V_{m(M_{\underline{i+1}})}) \\ &= 2n + 1 - i(\underline{i+1}). \end{aligned}$$

And $r = 2n+1-2i(\underline{i+1})$. This is because we have 1 contributing to the rank from the middle row and then we have $2n$ possible more to contribute to the rank r. For each row with a white checker, we do not have that basis vector nor do we have its mirror pair contributing to the rank, a total non-contribution of $2i(\underline{i+1})$. Then noting that $r = 2q+1$, we can conclude that $q = n - i(\underline{i+1})$. From the following calculation, we get that $n = i$.

$$k - i = q + 1$$

$$2n + 1 - i(\underline{i+1}) - i = n - i(\underline{i+1}) + 1$$

$$2n - i = n$$

$$n - i = 0$$

$$n = i$$

But $i < n$ (recall that we're finding M_{i+1} which is at largest M_n so $i+1 \le n$ and thus $i < n$) so we have a contradiction. Thus $rank(\overline{B}) = 2(k-i) - r \neq 1$ so $rank(\overline{B}) \geq 3$. □

4.3.2 Spaces built on T

Let inf $\in Q_\circ$. inf is an important element of $Q_\circ$ for the proof of theorem 4.4.1. We will define precisely which element of $Q_\circ$ is named inf for each case individually.

Definition 15. For a fixed point $t = ((V_m)_{m \in Q_\circ}, M_., F_{\leq c}) \in T$, choose F_{c+1} such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{\text{inf}} \subset F_{c+1}^\perp$

For fixed $t \in T$ let Q_t be the set of all such F_{c+1}.

Note that if $F_c \subset F_{c+1}$ and F_{c+1} is isotropic then the condition $F_{c+1} \subset F_c^\perp$ is met automatically.

Definition 16. $Q = \{(t, F_{c+1}) | t \in T, F_{c+1} \in Q_t\}$. Q is fibered over T with fibers Q_t.

For $(t, F_{c+1}) \in Q$, we have the following lemma.

Lemma 4.3.1. $(F_c^\perp \cap M_{d_E - 1}) \subset F_{c+1}^\perp \iff \dim(F_{c+1} \cap M_{d_W + 1}) \geq 1$

Proof. $F_c^\perp \cap M_{d_E-1} \subset F_{c+1}^\perp$ implies that $F_{c+1} \subset F_c + M_{d_E-1}^\perp = F_c + M_{d_W+1}$. By construction of T, F_c is transverse to the $M.$ flag, so $F_c \cap M_{d_W+1} = \langle 0 \rangle$. Thus we can write $F_{c+1} = F_c + L$ where L is a line in M_{d_W+1}, and we have $\dim(F_{c+1} \cap M_{d_W+1}) = 1$.

Now, suppose $\dim(F_{c+1} \cap M_{d_W+1}) = 1$. By construction of T, we know $F_c \cap M_{d_W+1} = \langle 0 \rangle$. This implies that there is a line $L \subset M_{d_W+1}$ such that $F_{c+1} = F_c + L$. So $F_{c+1} = (F_c + L) \subset (F_c + M_{d_W+1})$. Equivalently, $(F_c + M_{d_W+1})^\perp \subset F_{c+1}^\perp$. And

$$(F_c + M_{d_W+1})^\perp = (F_c^\perp \cap M_{d_W+1}^\perp) = (F_c^\perp \cap M_{d_E-1})$$

thus $F_c^\perp \cap M_{d_E-1} \subset F_{c+1}^\perp$. ∎

Definition 17. For a fixed point $t = ((V_m)_{m \in Q_\circ}, M., F_{\leq c}) \in T$, choose F_{c+1} such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{\inf} \subset F_{c+1}^\perp$
4. $V_{m(F_{c+1})} \subset F_{c+1} \subset V_a^\perp$

For fixed $t \in T$ let S_t be the set of all such F_{c+1}.

Definition 18. $W_\circ = \{(t, F_{c+1}) | t \in T, F_{c+1} \in S_t\}$. $W_\circ$ is fibered over T with fibers S_t.

Definition 19. For a fixed point $t = ((V_m)_{m \in Q_\circ}, M., F_{\leq c}) \in T$, choose F_{c+1} such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{\inf} \subset F_{c+1}^\perp$
4. $(F_c^\perp \cap M_{d_E-1}) \subset F_{c+1}^\perp$
5. $(F_c^\perp \cap M_{d_E+1}) \not\subset F_{c+1}^\perp$ (this is an open condition)

For fixed $t \in T$ let R_t be the set of all such F_{c+1}.

Definition 20. $W_{\bullet\bullet_{next}} = \{(t, F_{c+1}) | t \in T, F_{c+1} \in R_t\}$. $W_{\bullet\bullet_{next}}$ is fibered over T with fibers R_t.

Lemma 4.3.2. *$W_{\bullet\bullet_{next}}$ has a natural immersion*

$$W_{\bullet\bullet_{next}} \hookrightarrow OBS(Q_\circ) \times (X_\bullet \cup X_{\bullet_{next}})$$

Proof. Given a point $((V_m)_{m \in Q_\circ}, M_., F_{\leq c+1}) \in W_{\bullet\bullet_{next}}$, complete the $F_{\leq (c+1)}$ flag as follows: For F_j, $c+2 \leq j \leq n$, column j has a black checker in row r_j. Let $F_j = (M_{r_j} \cap F_{c+1}^{\perp}) + F_{c+1}$. Note that

1. F_j is isotropic. This is because F_{c+1} is isotropic and $F_{c+1}^{\perp} \cap M_{r_j}$ is isotropic because M_{r_j} is isotropic since $r_j \leq n$ for $c+2 \leq j \leq n$. In addition, $(F_{c+1}^{\perp} \cap M_{r_j}) \subset F_{c+1}^{\perp}$ so every vector in $F_{c+1}^{\perp} \cap M_{r_j}$ is orthogonal to every vector in F_{c+1}.

2. F_j has dimension j. There are two cases:

   (a) $r_j < r_{c+1}$. Then $\dim(F_{c+1}^{\perp} \cap M_{r_j}) = j - (c+1)$ and $(F_{c+1}^{\perp} \cap M_{r_j}) \cap F_{c+1} = \langle 0 \rangle$ so

   $$\begin{aligned} \dim(F_j) &= \dim(F_{c+1}^{\perp} \cap M_{r_j}) + \dim(F_{c+1}) - \dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1}) \\ &= j - (c+1) + (c+1) - 0 \\ &= j. \end{aligned}$$

   (b) $r_j > r_{c+1}$. Then $\dim(F_{c+1}^{\perp} \cap M_{r_j}) = j - c$ since the black checker in column $c+1$ is now included in the dimension count. This also means $\dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1}) = 1$. So

   $$\begin{aligned} \dim(F_j) &= \dim(F_{c+1}^{\perp} \cap M_{r_j}) + \dim(F_{c+1}) - \dim(F_{c+1}^{\perp} \cap M_{r_j} \cap F_{c+1}) \\ &= (j - c) + (c+1) - 1 \\ &= j. \end{aligned}$$

□

4.4 $W_\circ \cap W_{\bullet\bullet_{next}} = \mathrm{Cl}_{\mathrm{OBS}(\mathrm{Q}_\circ)\times(\mathrm{X}_\bullet \cup \mathrm{X}_{\bullet_{\mathrm{next}}})} \mathrm{X}_{\circ\bullet}$

Theorem 4.4.1. *We have the scheme-theoretic equality*

$$W_\circ \cap W_{\bullet\bullet_{next}} = \mathrm{Cl}_{\mathrm{OBS}(\mathrm{Q}_\circ)\times(\mathrm{X}_\bullet \cup \mathrm{X}_{\bullet_{\mathrm{next}}})} \mathrm{X}_{\circ\bullet}$$

Proof of theorem 4.4.1 depends on the type of move. We will prove theorem 4.4.1 for s_i moves with $d_E < n$ in section 4.4.1 and for s_0 moves in section 4.4.3. All proofs assume there is no white checker in column $c+1$ (recall remark in section 4.3.1).

In each case, the strategy of the proof is this: We fix an irreducible component Z of $W_\circ \cap W_{\bullet\bullet_{next}}$ and describe an open subscheme of Z explicitly as a tower of projective and quadric bundles over the dense open subset $OBS(Q_\circ)_\emptyset$. Through a description of the open subscheme of Z as a tower of bundles, we will show that Z has the expected codimension. Note that $W_\circ \cap W_{\bullet\bullet_{next}}$ is irreducible because T is irreducible and the fibers over $OBS(Q_\circ)_\emptyset$ are equidimensional. Since $W_\circ \cap W_{\bullet\bullet_{next}}$ is irreducible and, as we will show, Z is a component of the same dimension, we get that Z must be unique and thus $Z = W_\circ \cap W_{\bullet\bullet_{next}}$.

The following definition will play a large role in the proof of theorem 4.4.1.

Definition 21. The *expected codimension* over Q of $W_\circ \cap W_{\bullet\bullet_{next}}$ is

$$\mathrm{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \mathrm{codim}_Q(W_\circ) + \mathrm{codim}_Q(W_{\bullet\bullet_{next}})$$

4.4.1 s_i move with $d_E < n$

In this section, there is a white checker in either row d_E, row d_W, or both. Fix an irreducible component Z of $W_\circ \cap W_{\bullet\bullet_{next}}$. Necessarily, we have $\mathrm{codim}_Q Z \leq \mathrm{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$. We will show that Z is unique and $\mathrm{codim}_Q Z = \mathrm{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$.

The reader may wish to refer to Figures 4.3, 4.4, 4.5, and 4.8 as examples. Let x be the white checker on the western end of the northernmost diagonal of $Q_\circ$ whose eastern end is in a row greater than or equal to d_E and in a column greater than or equal to $\underline{c}$. Let d be the checker on the eastern end of the diagonal.

Define $R =$ (row of checker d) $- 1$. Let $\inf = x, a = m(F_{c+1}^{\perp}), x' = m(M_{R+1}), x'' = m(M_R)$, and $\sup = sup(x, x')$.

We have three cases: $(i)\inf = \mathbf{a}$, $(ii)\inf \neq \mathbf{a}, \mathbf{x}''$, and $(iii)\inf = \mathbf{x}''$. For the proofs of cases $(ii)\inf \neq \mathbf{a}, \mathbf{x}''$ and $(iii)\inf = \mathbf{x}''$, if there is a white checker in row d_E, then $R = d_E - 1$ and the proofs simplify in obvious ways.

Case (i) $\inf = \mathbf{a}$

In this case there are no white checkers in any columns from the column of x through $\underline{c+1}$. No white checkers in columns east of checker x through column $\underline{c+1}$ indicates that $x = a = m(F_{\underline{c+1}})$. To be in Q, $F_{\underline{c+1}}$ must contain V_x, so the $W_\circ$ condition that $F_{\underline{c+1}}$ also contain V_a is not a new condition. Thus $W_\circ = Q$ and we have $Z = W_{\bullet\bullet next}$. So Z is unique and

$$\operatorname{codim}_Q Z = \operatorname{codim}(W_{\bullet\bullet_{next}}) = \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$$

Case (ii) $\inf \neq \mathbf{a}, \mathbf{x}''$

For this case, there must be at least one white checker in a row directly north of row $R+1$ (the row of checker d) and at least one white checker in a column between the column of x and column $\underline{c+1}$. See Figures 4.3 and 4.4.

We will construct a dense open subscheme of Z. Let Z_V be the image of Z in $OBS(Q_\circ)$, Z_M be the image of Z in $OBS(Q_\circ)\times\{M.\}$, and Z_F be the image of Z in $T \subset OBS(Q_\circ)\times\{M.\}\times\{F_{\leq c}\}$. See Diagram (4.4).

$$\begin{tikzcd}
Q \arrow[d, "\pi_F"'] & W_\circ \cap W_{\bullet\bullet_{next}} \arrow[l, hook'] \arrow[dl] & \xi_t \arrow[l] & p_F^{-1}(t) \arrow[l, hook', "codim=\ell_6"'] \arrow[r] & Z \arrow[d, "p_F"] \\
T \arrow[d, "\pi_M"'] & & \pi_M^{-1}(V.,M.) \arrow[ll] & p_M^{-1}(V.,M.) \arrow[l, hook', "codim=\ell_5"'] \arrow[r] & Z_F \arrow[d, "p_M"] \\
\{(V.,M.)\} \arrow[d, "\pi_V"'] & & \pi_V^{-1}(V.) \arrow[ll] & p_V^{-1}(V.) \arrow[l, hook', "codim=\ell_4"'] \arrow[r] & Z_M \arrow[d, "p_V"] \\
OBS(Q_\circ) & & & & Z_V \arrow[llll, hook', "codim=\ell_1"']
\end{tikzcd} \tag{4.4}$$

Let $\boxed{\ell_1} = \operatorname{codim}_{OBS(Q_\circ)} Z_V$. Z_V is contained in some closed stratum of codimension at most ℓ_1, which corresponds to a set S of at most ℓ_1 quadrilaterals of $Q_\circ$ (recall strata in section 3.1). Thus $\ell_1 \geq |S|$. If $|S| = \ell_1$, then Z_V *is* the stratum $OBS(Q_\circ)_S$.

We next consider the choices for $M.$ with the conditions described in section 4.3.1. Let $\boxed{\ell_4}$ be the codimension of Z_M in the fibration

$$\pi_V^{-1}(Z_V) \to Z_V$$

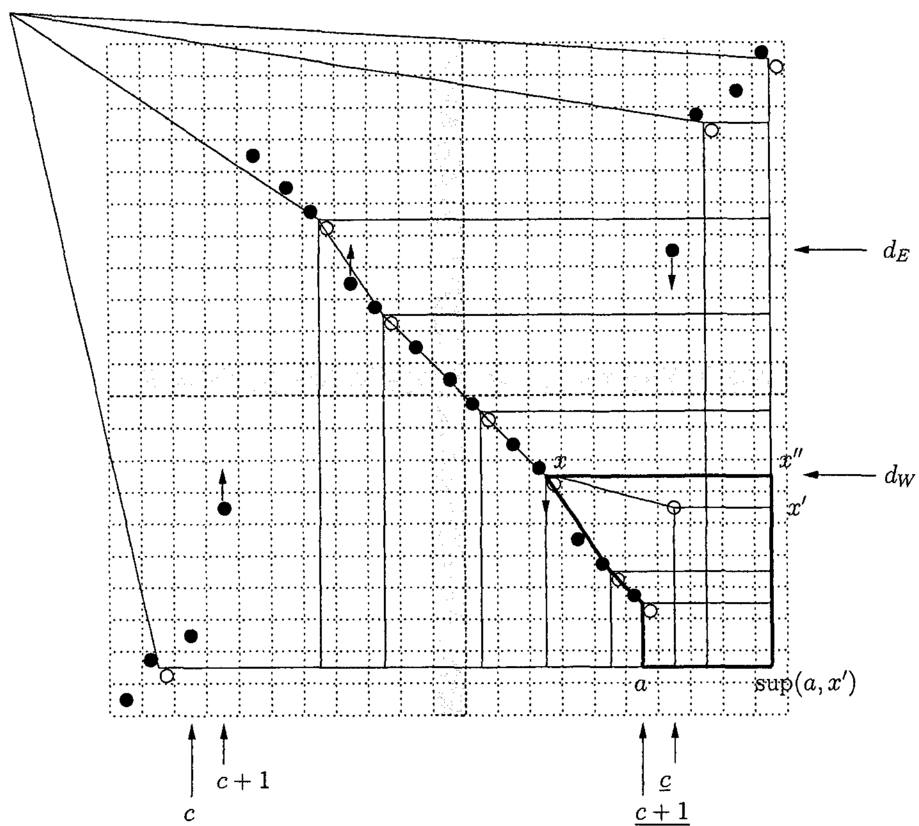


Figure 4.3: An example with $d_E < n$ with no white checker in column $c+1$. inf $\neq a, x''$. In this example, there is a white checker in row d_W but not row d_E.

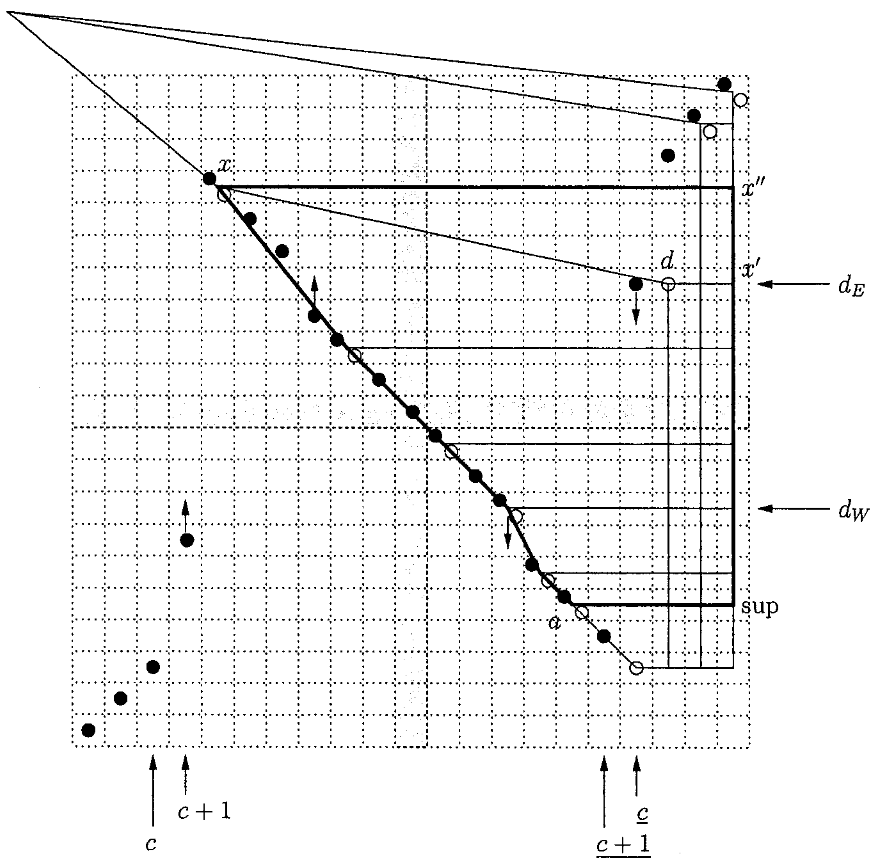


Figure 4.4: An example with $d_E < n$ with no white checker in column $c+1$. $\inf \neq a, x''$ and $R = d_E - 1$. In this example, there are white checkers in rows d_E and d_W.

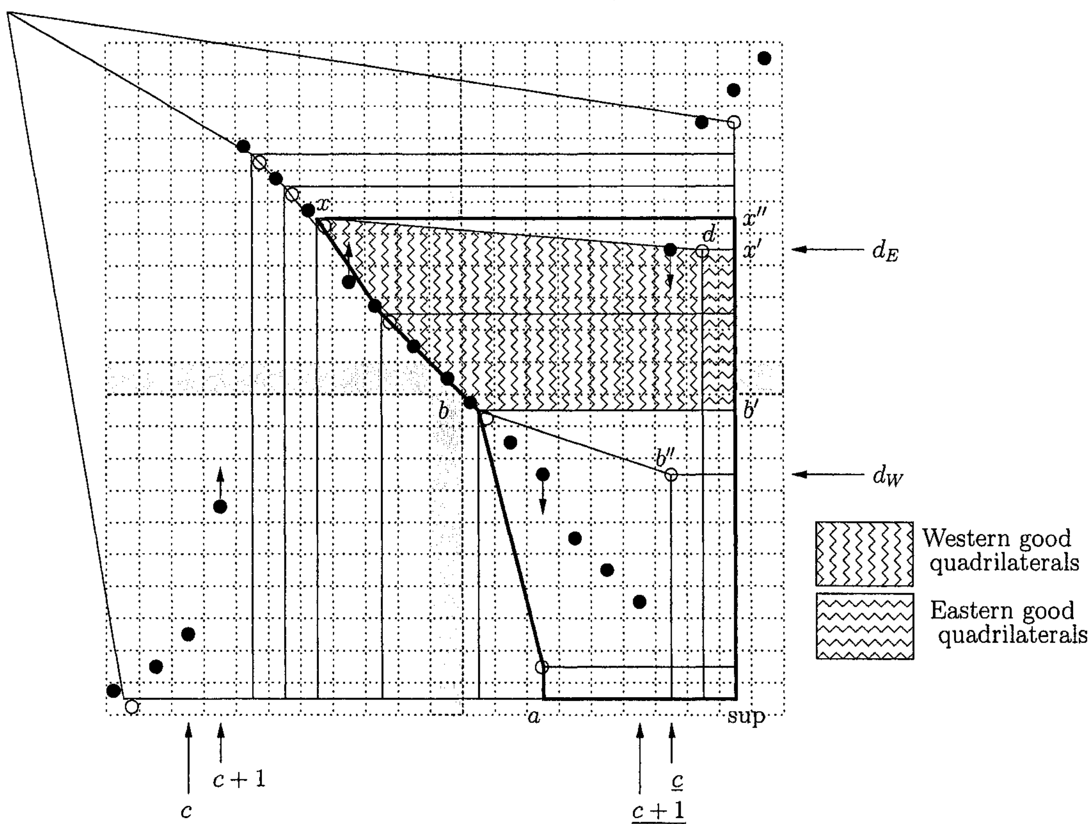


Figure 4.5: An example of an s_i move with a white checker in row $d_E < n$. The shaded regions and vertices b, b', and b'' are discussed in section 4.5

For a general point of Z, define

$$\boxed{\ell_2} = \dim(V_{\text{sup}} \cap M_{d_E-1}) - \dim(V_{x''} \cap M_{d_E-1}).$$

and

$$\boxed{\ell_2'} = \dim(V_{\text{sup}} \cap M_R) - \dim(V_{x''} \cap M_R).$$

By lemma 3.3.1 with R as above, $j = \dim(m(M_R)) = \dim x'', \delta = \dim(\text{sup}), B = Z_V$, and $B \to OFl(1, \ldots, \delta, 2n+1)$ the map giving the spaces of the northeast border of $OBS(Q_\circ)$, we have $\ell_4 \geq \ell_2'$.

By lemma 3.2.1, we also have $\ell_2' \geq \ell_2$, so

$$\ell_4 \geq \ell_2' \geq \ell_2.$$

Note that if there is a white checker in row d_E, then $R = d_E - 1$ and $\ell_2' = \ell_2$ immediately. Let $\boxed{\ell_5}$ be the codimension of Z_F in the fibration

$$\pi_M^{-1}(Z_M) \to Z_M$$

Then

$$\operatorname{codim}_T Z_F = \ell_1 + \ell_4 + \ell_5.$$

For a general point $t = (V., M., F_{\leq c}) \in Z_F$, consider the set $\{F_{c+1}\}$ where

1. $F_c \subset F_{c+1} \subset F_c^\perp$

2. $\dim(F_{c+1}) = c + 1$

3. F_{c+1} is isotropic

4. $V_{m(F_{c+1})} \subset F_{c+1} \subset V_a^\perp$

5. $F_{c+1} \subset (F_c^\perp \cap M_{d_E-1})^\perp$

Note that no white checker in column $c+1$ implies that the containment $V_{m(F_{c+1})} \subset F_{c+1}$ is already satisfied. Call this space ξ_t. The dimension of ξ_t is calculated here.

$$\dim \xi_t = \dim(V_a^\perp \cap (F_c^\perp \cap M_{d_E-1})^\perp) - \dim(F_c) - 1 - 1$$

The first -1 is for the isotropy condition and the second -1 is for choosing a line in $V_a^\perp \cap (M_{d_E-1} \cap F_c^\perp)^\perp$. The isotropy condition is a nontrivial condition since $V_a^\perp \cap (M_{d_E-1} \cap F_c^\perp)^\perp$ is not isotropic by Corollary 3.2.1. $V_a^\perp \cap (M_{d_E-1} \cap F_c^\perp)^\perp$ satisfies the hypotheses for Corollary 3.2.1 because $d_E - 1 < n$ so $M_{d_E-1} \cap F_c^\perp$ is isotropic. And $V_{\text{inf}} \subset V_a$ so $V_a^\perp \subset V_{\text{inf}}^\perp$. Thus $F_{c+1} \subset V_a^\perp$ implies $F_{c+1} \subset V_{\text{inf}}^\perp$. And $F_c \subset F_{c+1}$ with F_{c+1} isotropic implies that $F_{c+1} \subset F_c^\perp$. So we have

$$\begin{aligned}
\dim \xi_t &= \dim(V_a^\perp \cap (F_c^\perp \cap M_{d_E-1})^\perp) - c - 2 \\
&= \dim((V_a + (F_c^\perp \cap M_{d_E-1}))^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_a + (F_c^\perp \cap M_{d_E-1})) - c - 2 \\
&= 2n + 1 - (\dim V_a + \dim(F_c^\perp \cap M_{d_E-1}) - \dim(V_a \cap F_c^\perp \cap M_{d_E-1})) - c - 2 \\
&= 2n + 1 - \dim V_a - \dim(F_c^\perp \cap M_{d_E-1}) + \dim(V_a \cap M_{d_E-1}) - c - 2.
\end{aligned}$$

A note on the last line: $V_a = V_{m(F_{c+1}^\perp)} \subset F_{c+1}^\perp \subset F_c^\perp$. We now derive an expression for $\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$.

$$\begin{aligned}
\dim Q_t &= \dim(V_x^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_x) - c - 2. \\
\dim(W_\circ)_t &= 2n + 1 - \dim(V_a) - c - 2. \\
\operatorname{codim}_Q(W_\circ) &= \dim(Q_t) - \dim(W_\circ)_t \\
&= \dim(V_a) - \dim(V_x).
\end{aligned}$$

$$\begin{aligned}
\dim(W_{\bullet\bullet_{next}})_t &= \dim(V_x^\perp \cap (F_c^\perp \cap M_{d_E-1})^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_x) - \dim(F_c^\perp \cap M_{d_E-1}) + \dim(V_x \cap M_{d_E-1}) - c - 2. \\
\operatorname{codim}_Q(W_{\bullet\bullet_{next}}) &= \dim(Q_t) - \dim(W_{\bullet\bullet_{next}})_t \\
&= \dim(F_c^\perp \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1}).
\end{aligned}$$

So

$$\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \dim(V_a) - \dim V_x + \dim(F_c^\perp \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1}). \quad (4.5)$$

Now, $\operatorname{codim}_{Q_t} \xi_t$ is

$$\begin{aligned}
\operatorname{codim}_{Q_t} \xi_t &= \dim Q_t - \dim \xi_t \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim Q_t - \dim \xi_t - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - [\dim(V_a \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1})] \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - \ell_3
\end{aligned}$$

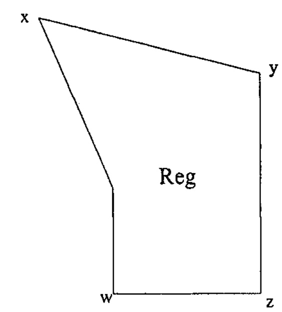


Figure 4.6: A general outline of the region defined by $\mathbf{xyzw}$.

where

$$\boxed{\ell_3} = \dim(V_a \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1}).$$

Since $V_x \subset V_a$, ℓ_3 is non-negative. Let $\boxed{\ell_6}$ be the codimension of the fiber $p_F^{-1}(t) \subset Z \to Z_F$ in ξ_t. See Diagram (4.4). Then $\operatorname{codim}_{Q_t}(p_F^{-1}(t)) = \operatorname{codim}_{Q_t} \xi_t + \ell_6$ and we have

$$\begin{aligned}
\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) &= (\ell_1 + \ell_4 + \ell_5 + \operatorname{codim}_{Q_t}(p_F^{-1}(t))) - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \ell_1 + \ell_4 + \ell_5 + \operatorname{codim}_{Q_t} \xi_t + \ell_6 - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \ell_1 + \ell_4 + \ell_5 + (\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - \ell_3) \\
&\quad + \ell_6 - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \ell_1 + \ell_4 + \ell_5 + \ell_6 - \ell_3.
\end{aligned}$$

Now, $\ell_5 \geq 0$ and $\ell_6 \geq 0$ because these are codimensions. And $\ell_4 \geq \ell_2$ by lemma 3.3.1. So we have

$$\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3.$$

Since Z is a component of $W_\circ \cap W_{\bullet\bullet_{next}}$, it must be true that $\operatorname{codim} Z \leq \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$. Thus $\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \leq 0$ (nonpositive). We will show that $\ell_1 + \ell_2 - \ell_3 \geq 0$, which will force $\operatorname{codim} Z = \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$.

Consider the region defined by $\mathbf{xx}'' \sup \mathbf{a}$.

Definition 22. Name the northwest, northeast, southeast, and southwest vertices $\mathbf{x}, \mathbf{y}, \mathbf{z}$, and $\mathbf{w}$ respectively. See Figure 4.6. The region defined by $\mathbf{xyzw}$ has boundary edges defined as follows: For the northern boundary (between $\mathbf{x}$ and $\mathbf{y}$), choose the southern most path from $\mathbf{y}$ to $\mathbf{x}$ such that if $\mathbf{m} \in Q_\circ$ is a node on the path, then $\mathbf{x} \prec \mathbf{m} \prec \mathbf{y}$. For the eastern boundary (between $\mathbf{z}$ and $\mathbf{y}$), choose the western most path from $\mathbf{y}$ to $\mathbf{z}$ such that if $\mathbf{m} \in Q_\circ$ is a node on the path, then $\mathbf{y} \prec \mathbf{m} \prec \mathbf{z}$. Similarly, the southern boundary is the northern most path between $\mathbf{z}$ and $\mathbf{w}$ and the western boundary is the eastern most path between $\mathbf{x}$ and $\mathbf{w}$.

The *total content* of the quadrilaterals in this region is the sum over the content of all quadrilaterals in the region. This is a linear combination of the labels of the vertices (see equation (3.5)). The net contribution of a vertex $\mathbf{m} \in Q_\circ$ is the number of quadrilaterals in the region of which it is the northeast or southwest corner, minus the number of which it is the northwest or southeast corner. Hence the only non-zero contribution to the total content is the following (see also [22]):

- Any internal diagonal edge contributes the label of its larger edge minus the label of its smaller edge (a non-negative contribution).
- The northeast and southwest corner vertices contribute their labels and the northwest and southeast corner vertices contribute the negative of their labels.

For the region defined by $\mathbf{x}\mathbf{x}''\sup\mathbf{a}$, label vertex $\mathbf{m}$ of $Q_\circ$ with the value $\dim(V_m \cap M_{d_E-1})$. Then the total content of this region is

$$\begin{aligned} TC &= (\text{internal diagonal contribution}) + \dim(V_a \cap M_{d_E-1}) + \dim(V_{x''} \cap M_{d_E-1}) \\ &\qquad - \dim(V_x \cap M_{d_E-1}) - \dim(V_{\sup} \cap M_{d_E-1}) \\ &\geq -(\dim(V_{\sup} \cap M_{d_E-1}) - \dim(V_{x''} \cap M_{d_E-1})) + (\dim(V_a \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1})) \\ &= -\ell_2 + \ell_3. \end{aligned}$$

The content is bounded above by $|S|$ (lemma 3.4.2) which in turn is bounded above by ℓ_1 and we have

$$\ell_1 \geq |S| \geq -\ell_2 + \ell_3$$

which implies

$$\ell_1 + \ell_2 - \ell_3 \geq 0.$$

So we have equality in all inequalities. In particular, $\ell_5 = \ell_6 = 0$, $\ell_4 = \ell_2' = \ell_2$, the internal diagonal contribution is zero, and $\ell_1 = |S| =$ total content.

Next we use lemma 3.4.1(b) but first must confirm that the hypotheses for this lemma hold.

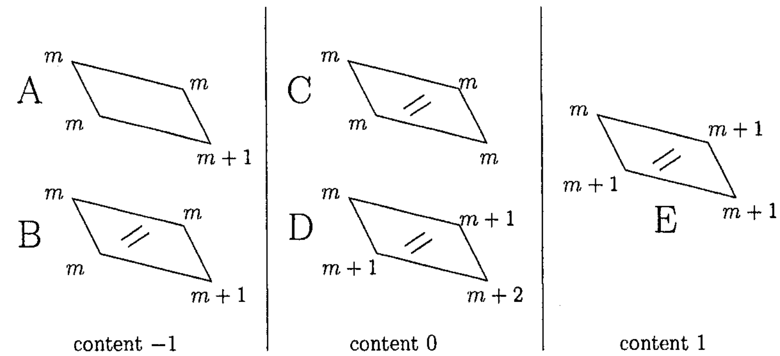


Figure 4.7: The five types of quadrilaterals we are interested in for lemma 4.4.1.

Lemma 4.4.1. *If* $\ell_1 = |S|$ *then the hypotheses for lemma 3.4.1(b) are satisfied. In particular, there are no negative-content quadrilaterals and there are no zero-content quadrilaterals with "=" signs.*

Proof. We know $\ell_1 =$ total content and $|S| = \#$ of quads with "=" signs and we know $\ell_1 = |S|$. From lemma 3.4.1, there are five types of quadrilaterals that we are interested in. See Figure 4.7.

$$\ell_1 = \text{total content} = (-1)|A| + (-1)|B| + |E|$$

where $|X| = \#$ of quads of type X. And

$$|S| = \#\text{ of quads with "=" signs} = |B| + |C| + |D| + |E|.$$

Now, $\ell_1 = |S|$ implies

$$\begin{aligned}(-1)|A| + (-1)|B| + |E| &= |B| + |C| + |D| + |E| \\ \Rightarrow 0 &= |A| + 2|B| + |C| + |D|\end{aligned}$$

Which implies $|A| = |B| = |C| = |D| = 0$. In particular, $|A| = |B| = 0$ means there are no negative-content quadrilaterals, and $|C| = |D| = 0$ means there are no zero-content quadrilaterals with "=" signs. So the hypotheses for lemma 3.4.1 are satisfied. □

Since all internal diagonals have the same label on both vertices, edge **xd** has the same label on both vertices. Using lemma 3.4.1$(b)(i)$, we move south from **xd** and conclude that $\dim(V_x \cap M_{d_E-1}) = \dim(V_a \cap M_{d_E-1})$. So $\ell_3 = 0$.

We also have that $\ell_4 = \ell_2' = \ell_2$. We will show that $\ell_2' = 0$. Consider the same region, defined by $\mathbf{xx}''\sup\mathbf{a}$, but now label vertex $\mathbf{m} \in Q_\circ$ with $\dim(V_m \cap M_R)$ for a general point of Z. The total content, TC_R of the region is

$$\begin{aligned}TC_R &= (\text{internal diagonal contribution})_R + \dim(V_a \cap M_R) + \dim(V_{x''} \cap M_R)\\ &\quad - \dim(V_x \cap M_R) - \dim(V_{\sup} \cap M_R)\\ &\geq -(\dim(V_{\sup} \cap M_R) - \dim V_{x''} \cap M_R) + (\dim(V_a \cap M_R) - \dim(V_x \cap M_R))\\ &= -\ell_2' + \ell_3'\end{aligned}$$

where $\boxed{\ell_3'} = \dim(V_a \cap M_R) - \dim(V_x \cap M_R)$. In the case where $R = d_E - 1$, we have $TC = TC_R$.

By lemma 3.2.1, we have $\ell_3' \geq \ell_3$. The total content is still bounded above by ℓ_1 so we have

$$\ell_1 \geq -\ell_2' + \ell_3' \geq -\ell_2' + \ell_3 = -\ell_2 + \ell_3. \tag{4.6}$$

We already have equality in $\ell_1 = -\ell_2 + \ell_3$, so we get equality across (4.6) as well. In particular, $\ell_3' = \ell_3 = 0$ and the internal diagonal contribution is zero.

Moving east from edge $\mathbf{xd}$ and using lemma 3.4.1$(b)(ii)$, we get that

$$\dim(V_{x'} \cap M_R) = \dim(V_{x''} \cap M_R) = \dim(V_{x''}) < \dim(V_{x'}).$$

This implies that $V_{x'} \not\subset M_R$. By lemma 3.3.1, with equality $\ell_2' = \ell_4$ and $V_{x'} \not\subset M_R$, we must have $\ell_2' = 0$. Thus $\ell_4 = \ell_2' = \ell_2 = 0$ and we have $0 = \ell_1 + \ell_2 - \ell_3 = \ell_1$. And

$$\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \ell_1 + \ell_4 + \ell_5 + \ell_6 - \ell_3 = 0.$$

For this case, we have described an open subscheme of Z explicitly as a tower of projective and quadric bundles over $OBS(Q_\circ)_\emptyset$. Thus Z is unique and we've shown that Z has the expected codimension.

Case (iii) $\inf = \mathbf{x}''$

This case occurs if and only if there are no white checkers directly above row $R+1$ in columns $\underline{c}$ or greater. See Figure 4.8 as an example. The argument for the case $\inf \neq \mathbf{a}, \mathbf{x}''$ applies verbatim until we conclude that $0 \geq \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3$. Consider the region defined by vertices $\mathbf{x} = \inf = \mathbf{x}'', \mathbf{x}', \sup$, and $\mathbf{a}$. Label vertex $\mathbf{m} \in Q_\circ$ with $\dim(V_m \cap M_{d_E-1})$. Now, $V_{x''}$ is a hyperplane in $V_{x'}$ so

$$\dim(V_{x'} \cap M_{d_E-1}) - \dim(V_{x''} \cap M_{d_E-1}) = \epsilon$$

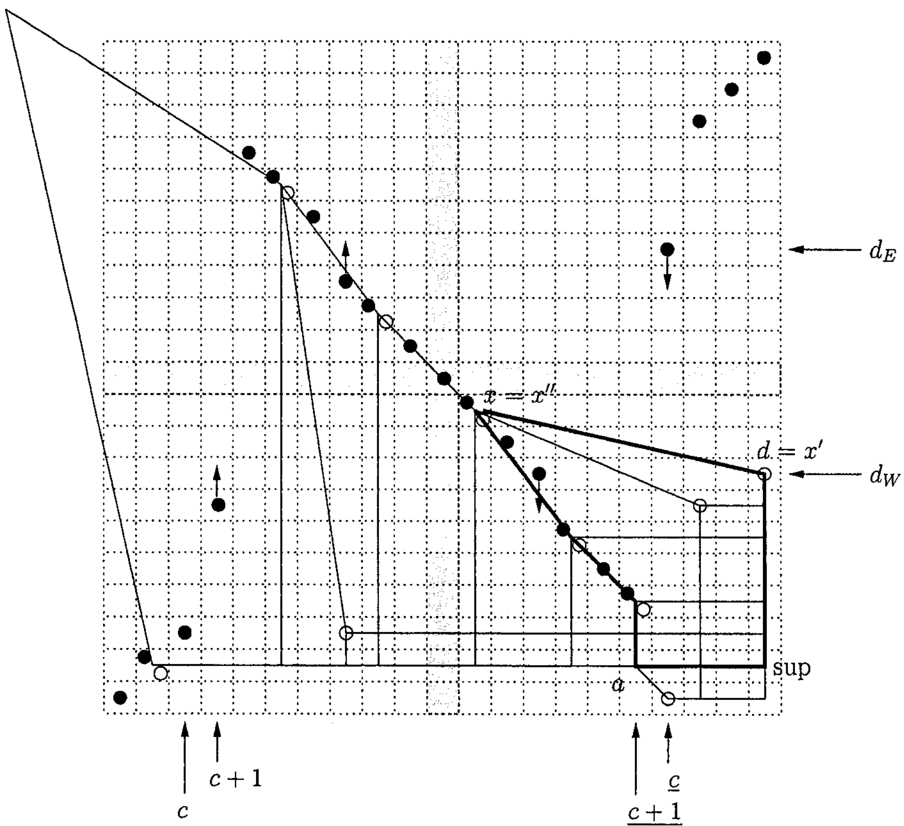


Figure 4.8: $d_E < n$ with no white checker in column $c+1$. inf $= \mathbf{x}''$ and $R = d_W - 1$.

where $\epsilon = 0$ or 1. The total content of region $\mathbf{x}\mathbf{x}'\sup\mathbf{a}$ is

$$\begin{aligned}
TC &= (\text{internal diagonal contribution}) + \dim(V_a \cap M_{d_E-1}) + \dim(V_{x'} \cap M_{d_E-1}) \\
&\quad - \dim(V_x \cap M_{d_E-1}) - \dim(V_{\sup} \cap M_{d_E-1}) \\
&= (\text{internal diagonal contribution}) + \dim(V_a \cap M_{d_E-1}) + (\epsilon + \dim(V_{x''} \cap M_{d_E-1})) \\
&\quad - \dim(V_x \cap M_{d_E-1}) - \dim(V_{\sup} \cap M_{d_E-1}) \\
&\geq \epsilon + (\dim(V_a \cap M_{d_E-1}) - \dim(V_x \cap M_{d_E-1})) - (\dim(V_{\sup} \cap M_{d_E-1}) - \dim(V_{x''} \cap M_{d_E-1})) \\
&= \epsilon + \ell_3 - \ell_2.
\end{aligned}$$

This is bounded above by ℓ_1 so $\ell_1 \geq \epsilon + \ell_3 - \ell_2$ and thus $\ell_1 + \ell_2 - \ell_3 \geq \epsilon$. So we have $0 \geq \ell_1 + \ell_2 - \ell_3 \geq \epsilon$ and so $\epsilon = 0$, and equality holds in all previous inequalities. In particular, $\ell_4 = \ell_2' = \ell_2$ and the internal diagonal contribution is zero. Using lemma 3.4.1$(b)(i)$ and moving south from edge $\mathbf{xd}$, we have $\ell_3 = 0$. We now show that $\ell_2' = 0$. Relabel region $\mathbf{x}\mathbf{x}'\sup\mathbf{a}$ with labels $\dim(V_m \cap M_R)$. The new total content, TC_R is

$$\begin{aligned}
TC_R &= (\text{internal diagonal contribution})_R + \dim(V_a \cap M_R) + \dim(V_{x'} \cap M_R) \\
&\quad - \dim(V_x \cap M_R) - \dim(V_{\sup} \cap M_R) \\
&= (\text{internal diagonal contribution})_R + \dim(V_a \cap M_R) + (\epsilon_R + \dim(V_{x''} \cap M_R)) \\
&\quad - \dim(V_x \cap M_R) - \dim(V_{\sup} \cap M_R) \\
&\geq \epsilon_R + (\dim(V_a \cap M_R) - \dim(V_x \cap M_R)) - (\dim(V_{\sup} \cap M_R) - \dim(V_{x''} \cap M_R)) \\
&= \epsilon_R + \ell_3' - \ell_2'
\end{aligned}$$

where $\epsilon_R = \dim(V_{x'} \cap M_R) - \dim(V_{x''} \cap M_R) \geq 0$. If there is a white checker in row d_E then $R = d_E - 1$ and $TC_R = TC$. By lemma 3.2.1 we have $\ell_3' \geq \ell_3$.

The total content of the region is bounded above by ℓ_1 so

$$\ell_1 \geq -\ell_2' + \ell_3' + \epsilon_R \geq -\ell_2' + \ell_3' \geq -\ell_2' + \ell_3 = -\ell_2 + \ell_3. \tag{4.7}$$

We already have equality in $\ell_1 = -\ell_2 + \ell_3$ so we now have $\epsilon_R = 0, \ell_3' = \ell_3 = 0$, and the $(\text{internal diagonal contribution})_R$ is zero. Since $\epsilon_R = 0$, we have

$$\dim(V_{x'} \cap M_R) = \dim(V_{x''} \cap M_R) = \dim(V_{x''}) < \dim(V_{x'}).$$

This implies that $V_{x'} \not\subset M_R$. By lemma 3.3.1, with equality $\ell_2' = \ell_4$ and $V_{x'} \not\subset M_R$, we must have $\ell_2' = 0$. Thus $\ell_4 = \ell_2' = \ell_2 = 0$ and we have $0 = \ell_1 + \ell_2 - \ell_3 = \ell_1$. So for $\inf = \mathbf{x}''$ we have described an appropriate subscheme of Z and thus completed the proof in this case.

4.4.2 s_i move with $d_E \geq n+2$

There are partial results for the case of an s_i move with $d_E \geq n+2$. We omit them for this dissertation.

4.4.3 s_0 move

For a nontrivial s_0 move, there is a white checker in row n. Let $\inf = \inf(a, a'')$, $\mathbf{a} = m(F_{c+1}^{\perp})$, $\mathbf{a}' = m(M_n)$, $\mathbf{a}'' = m(M_{n-1})$, and $\sup = \sup(a, a')$. Figures 4.9 and 4.10 are examples of nontrivial s_0 moves. We have three cases.

Case (i) $\inf = \mathbf{a}$

If there are no white checkers in columns $n+2 \leq col \leq \underline{c+1}$ then $\inf = \mathbf{a}$. No white checkers in this region implies that $V_a \subset V_{\text{inf}} + F_c$ which means $V_{\text{inf}}^{\perp} \cap F_c^{\perp} \subset V_a^{\perp}$. We already know that all elements of Q satisfy the conditions $F_{c+1} \subset V_{\text{inf}}^{\perp}$ and $F_{c+1} \subset F_c^{\perp}$, so $F_{c+1} \subset V_a^{\perp}$ is not a new condition. Thus $W_\circ = Q$ and we have $Z = W_{\bullet\bullet_{next}}$. So Z is unique and $\text{codim}_Q Z = \text{codim}(W_{\bullet\bullet_{next}}) = \text{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$.

Case (ii) $\inf \neq \mathbf{a}, \mathbf{a}''$

See Figures 4.9 and 4.10. There must be at least one white checker in columns $n+2 \leq col \leq \underline{c+1}$, and at least one white checker in the region bounded by $1 \leq row < n$ and $\underline{c} \leq col \leq 2n+1$. We will construct a dense open subscheme of Z. Let Z_V, Z_M, and Z_F be described as in section 4.4.1. See Diagram (4.4).

Let $\boxed{\ell_1} = \text{codim}_{OBS(Q_\circ)} Z_V$. Z_V is contained in some closed stratum of codimension at most ℓ_1 which corresponds to a set S of at most ℓ_1 quadrilaterals of $Q_\circ$ (recall strata in section 3.1). Thus $\ell_1 \geq |S|$. If $|S| = \ell_1$ then Z_V *is* the stratum $OBS(Q_\circ)_S$.

We next consider the choices for $M.$ with the conditions described in section 4.3.1. Let $\boxed{\ell_4}$ be the codimension of Z_M in the fibration

$$\pi_V^{-1}(Z_V) \to Z_V.$$

Define for a general point of Z,

$$\begin{aligned} \boxed{\ell_2} &= \dim(V_{\text{sup}} \cap M_{n-1}) - \dim(V_{m(M_{n-1})}) \\ &= \dim(V_{\text{sup}} \cap M_{n-1}) - \dim(V_{a''}). \end{aligned}$$

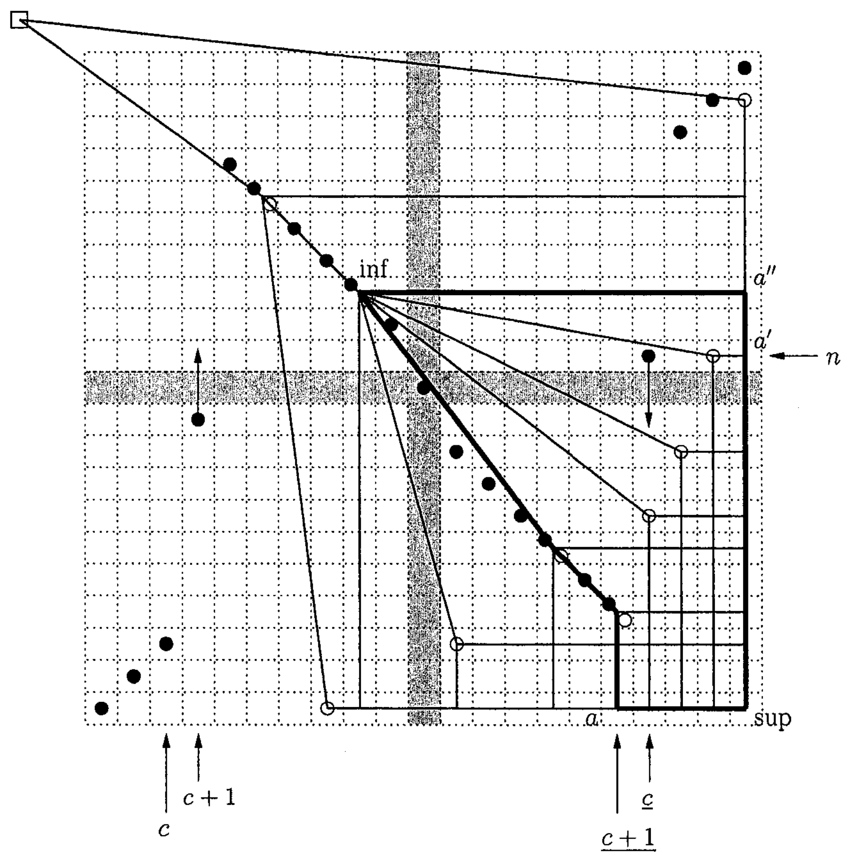


Figure 4.9: Example of an s_0 move where there is no white checker in column $c+1$ and there is a white checker in row n. $\inf(a, a'') \neq a, a''$

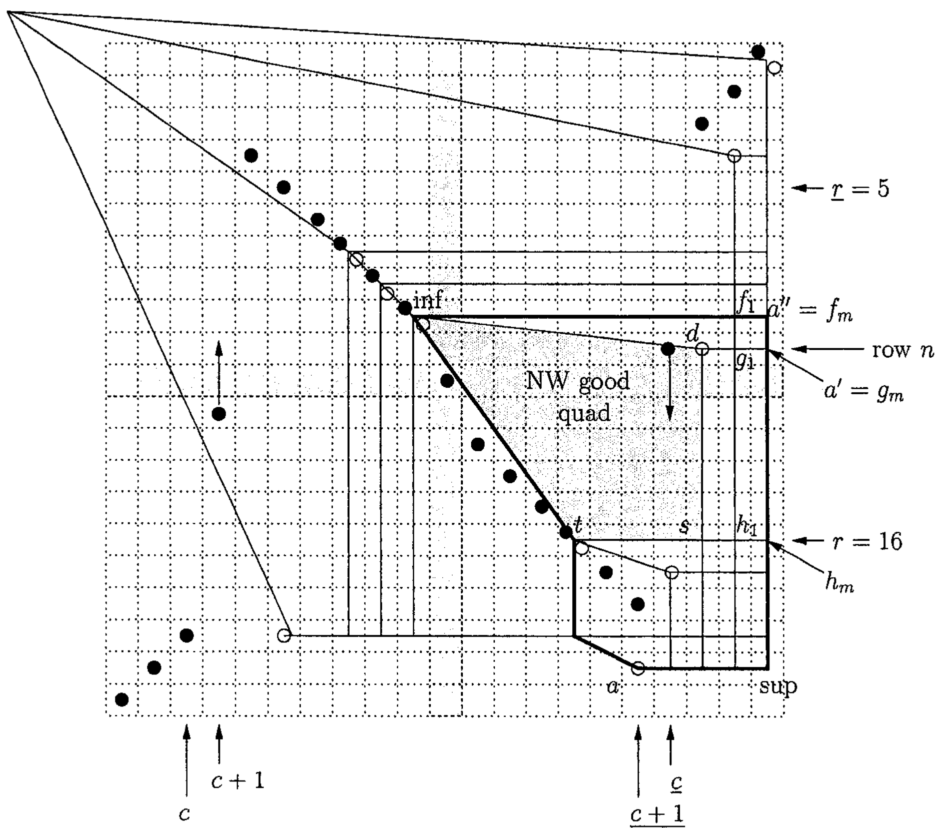


Figure 4.10: Example of a nontrivial s_0 move. Labels r, s, t, f_i, g_i, and h_i are discussed in section 4.7.3.

By lemma 3.3.1, taking $R = n-1$, $j = \dim(m(M_{n-1})) = \dim(a'')$, $\delta = \dim(\sup(a,a'))$, $B = Z_V$, and $B \to OFl(1,\ldots,\delta,2n+1)$ the map giving the spaces of the northeast border of $OBS(Q_\circ)$, we have $\ell_4 \geq \ell_2$.

Let $\boxed{\ell_5}$ be the codimension of Z_F in the fibration

$$\pi_M^{-1}(Z_M) \to Z_M.$$

Then we have

$$\operatorname{codim}_T Z_F = \ell_1 + \ell_4 + \ell_5.$$

For a general point $t = (V_., M_., F_{\leq c}) \in Z_F$, consider the set $\{F_{c+1}\}$ where

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} isotropic
3. $V_{m(F_{c+1})} \subset F_{c+1} \subset V_a^\perp$
4. $F_{c+1} \subset (F_c^\perp \cap M_{n-1})^\perp$

Call this space ξ_t. The dimension of ξ_t is calculated here:

$$\dim \xi_t = \dim(V_a^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - \dim(F_c) - 1 - 1.$$

We subtract 1 because we choose a line in the space $(V_a^\perp \cap (F_c^\perp \cap M_{n-1})^\perp)/F_c$ and we subtract 1 for isotropy. Here, V_a is isotropic and so is $F_c^\perp \cap M_{n-1}$ since M_{n-1} is isotropic, so by Corollary 3.2.1, $V_a^\perp \cap (F_c^\perp \cap M_{n-1})^\perp$ is not isotropic. Continuing the calculation:

$$\begin{aligned}
\dim \xi_t &= \dim(V_a^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - c - 2 \\
&= \dim((V_a + (F_c^\perp \cap M_{n-1}))^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_a + (F_c^\perp \cap M_{n-1})) - c - 2 \\
&= 2n + 1 - \dim(V_a) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_a \cap F_c^\perp \cap M_{n-1}) - c - 2 \\
&= 2n + 1 - \dim(V_a) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_a \cap M_{n-1}) - c - 2.
\end{aligned}$$

Note that for the last step of the calculation we have $V_a \subset V_{m(F_{\underline{c}})} \subset F_{\underline{c}} = F_c^\perp$.

Recall from Definition 21 that $\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \operatorname{codim}_Q W_\circ + \operatorname{codim}_Q W_{\bullet\bullet_{next}}$. So we need

$$\begin{aligned}
\dim(Q_t) &= \dim(V_{\text{inf}}^\perp) - c - 2 \\
&= 2n + 1 - \dim(V_{\text{inf}}) - c - 2.
\end{aligned}$$

$$\begin{aligned}
\dim(W_\circ)_t &= \dim(V_a^\perp \cap V_{\text{inf}}^\perp) - c - 2 \\
&= 2n+1-\dim(V_a) - c - 2. \\
\dim(W_{\bullet\bullet_{next}})_t &= \dim(V_{\text{inf}}^\perp \cap (F_c^\perp \cap M_{n-1})^\perp) - c - 2 \\
&= \dim((V_{\text{inf}} + (F_c^\perp \cap M_{n-1}))^\perp) - c - 2 \\
&= 2n+1-\dim(V_{\text{inf}} + (F_c^\perp \cap M_{n-1})) - c - 2 \\
&= 2n+1-\dim(V_{\text{inf}}) - \dim(F_c^\perp \cap M_{n-1}) \\
&\quad + \dim(V_{\text{inf}} \cap F_c^\perp \cap M_{n-1}) - c - 2 \\
&= 2n+1-\dim(F_c^\perp \cap M_{n-1}) - c - 2.
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{codim}_Q(W_\circ) &= \dim(Q_t) - \dim(W_\circ)_t \\
&= \dim(V_a) - \dim(V_{\text{inf}}). \\
\operatorname{codim}_Q(W_{\bullet\bullet_{next}}) &= \dim(F_c^\perp \cap M_{n-1}) - \dim(V_{\text{inf}}).
\end{aligned}$$

so

$$\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \dim(V_a) + \dim(F_c^\perp \cap M_{n-1}) - 2\dim(V_{\text{inf}}). \tag{4.8}$$

We now calculate $\operatorname{codim}_{Q_t} \xi_t$.

$$\begin{aligned}
\operatorname{codim}_{Q_t} \xi_t &= \dim Q_t - \dim \xi_t \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim Q_t - \dim \xi_t - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) + 2n+1-\dim(V_{\text{inf}}) - c - 2 \\
&\quad - (2n+1-\dim(V_a) - \dim(F_c^\perp \cap M_{n-1}) + \dim(V_a \cap M_{n-1}) - c - 2) \\
&\quad - (\dim(V_a) + \dim(F_c^\perp \cap M_{n-1}) - 2\dim(V_{\text{inf}})) \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - [\dim(V_a \cap M_{n-1}) - \dim(V_{\text{inf}})] \\
&= \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) - \ell_3
\end{aligned}$$

where $\boxed{\ell_3} = \dim(V_a \cap M_{n-1}) - \dim(V_{\text{inf}})$. Let $\boxed{\ell_6}$ be the codimension of the fiber $p_F^{-1}(t) \subset Z \to Z_F$ in ξ_t. See Diagram (4.4). Then $\operatorname{codim}_{Q_t}(p_F^{-1}(t)) = \operatorname{codim}_{Q_t} \xi_t + \ell_6$. And we have

$$\begin{aligned}
\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) &= \ell_1 + \ell_4 + \ell_5 + \operatorname{codim}_{Q_t} \xi_t + \ell_6 - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \ell_1 + \ell_4 + \ell_5 + \ell_6 - \ell_3.
\end{aligned}$$

Now, ℓ_5 and ℓ_6 are codimensions, so $\ell_5, \ell_6 \geq 0$. And $\ell_4 \geq \ell_2$ by lemma 3.3.1, so

$$\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3.$$

Since Z is a component of $W_\circ \cap W_{\bullet\bullet_{next}}$, it must be true that $\operatorname{codim} Z \leq \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}})$. So $0 \geq \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) \geq \ell_1 + \ell_2 - \ell_3$. We now show that $\ell_1 + \ell_2 - \ell_3 \geq 0$.

Label vertex $\mathbf{m}$ of $Q_\circ$ with the value $\dim(V_m \cap M_{n-1})$ for a general point of Z. So inf is labelled $\dim(V_{\inf})$ and $\mathbf{a}''$ is labelled $\dim(V_{a''})$. Consider the region defined by $\mathbf{x} = \inf, \mathbf{y} = \mathbf{a}'', \mathbf{z} = \sup$, and $\mathbf{w} = \mathbf{a}$ (see Definition 22). The total content of region $\inf \mathbf{a}'' \sup \mathbf{a}$ is:

$$\begin{aligned}
TC &= (\text{internal diagonal contribution}) + \dim(V_a \cap M_{n-1}) \\
&\quad + \dim(V_{a''} \cap M_{n-1}) - \dim(V_{\inf} \cap M_{n-1}) - \dim(V_{\sup} \cap M_{n-1}) \\
&\geq -(\dim(V_{\sup} \cap M_{n-1}) - \dim(V_{a''})) + (\dim(V_a \cap M_{n-1}) - \dim(V_{\inf})) \\
&= -\ell_2 + \ell_3.
\end{aligned}$$

The content is bounded above by $|S|$ (lemma 3.4.2) which in turn is bounded above by ℓ_1, so $\ell_1 \geq |S| \geq -\ell_2 + \ell_3$ and we have $\ell_1 + \ell_2 - \ell_3 \geq 0$. This means $0 = \operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \ell_1 + \ell_2 - \ell_3$ and we have equality on all inequalities. In particular, $Z_V = OBS(Q_\circ)_S$, $\ell_2 = \ell_4$, $\ell_5 = \ell_6 = 0$, the internal diagonal contribution is zero, and $\ell_1 = |S| =$ total content.

Internal diagonal contribution is zero implies that all internal diagonals have the same labels on either end. Let $\mathbf{d}$ be the vertex of the white checker in row n, then $\inf \mathbf{d}$ is an internal diagonal and inf and $\mathbf{d}$ have the same label: $\dim(\inf)$. By lemma 3.4.1$(b)(ii)$, we can deduce that $\mathbf{a}''$ and $\mathbf{a}'$ have the same label. So we have

$$\dim(V_{a'} \cap M_{n-1}) = \dim(V_{a''} \cap M_{n-1}) = \dim(V_{a''}) < \dim(V_{a'}).$$

So $V_{a'} \not\subset M_{n-1}$. By lemma 3.3.1, since we have the equality $\ell_4 = \ell_2$, it must be that either $\ell_2 = 0$ or $V_{j+1} \subset M_R$, i.e. $V_{a'} \subset M_{n-1}$, for all points in Z. Since this is not the case, $\ell_2 = 0$.

By lemma 3.4.1$(b)(i)$, we work our way south from the internal diagonal $\inf \mathbf{d}$ to conclude that inf and $\mathbf{a}$ have the same label. Namely, $\dim(V_{\inf}) = \dim(V_a \cap M_{n-1})$, so $\ell_3 = 0$. Which also gives us $\ell_1 = 0$.

Thus $\operatorname{codim} Z - \operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \ell_1 + \ell_2 - \ell_3 = 0$. So for the s_0-case: no white checker in column $c+1$, white checker in row n, $\inf \neq \mathbf{a}, \mathbf{a}''$, we have described an open subscheme of Z explicitly as a tower of projective and quadric bundles over $OBS(Q_\circ)_{\emptyset = S}$. Thus Z is unique and we've shown Z has the expected codimension.

Case (iii) $\inf = \mathbf{a}''$

The proof is verbatim the case in the s_i moves when there is a white checker in row $d_E < n$. The picture is a little bit different (there are only two black checkers moving) but otherwise similar. Replace d_E with n and the proof is equivalent.

4.5 Irreducible components of D_Q are a subset of the D_S's

We describe the components of D_Q in terms of strata of $OBS(Q_\circ)$. Theorem 4.5.1 is valid for the case of an s_i move where there is a white checker in row $d_E < n$ and the case of a nontrivial s_0 move. We give some definitions, state the theorem, then prove it for each case individually.

4.5.1 s_i move with $d_E < n$

There is a white checker in row d_E

Let $\mathbf{d}$ be the vertex of $Q_\circ$ where the white checker in row d_E is located. Define the *western good quadrilaterals* of $Q_\circ$ to be those quads with eastern two vertices dominating $\mathbf{d}$ and western two vertices dominated by $\mathbf{a} = m(F_{\underline{c+1}})$. Let the *eastern good quadrilaterals* be those quads whose vertices all dominate $\mathbf{d}$, and are east of a western good quad. Let $\mathbf{b}$ be the southwestern corner of the region of good quads and $\mathbf{b}'$ be the southeastern corner of the region of good quads. The region of good quads is $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}'\mathbf{b}\mathbf{b}'$ (possibly empty). It may be helpful to refer to Figure 4.5.

Define $W_{\bullet_{next}} \subset Q$, fibered over T, with fibers $\{F_{c+1}\}$ such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_{\inf} \subset F_{c+1}^\perp$
4. $(F_c^\perp \cap M_{d_E}) \subset F_{c+1}^\perp$
5. $(F_c^\perp \cap M_{d_E+1}) \not\subset F_{c+1}^\perp$ (this is an open condition)

In other words, $W_{\bullet_{next}}$ is the pullback of the Cartier divisor $X_{\bullet_{next}} \subset X_{\bullet\bullet_{next}}$ to $W_{\bullet\bullet_{next}}$. Let D_Q be the pullback of the Cartier divisor $X_{\bullet_{next}} \subset X_{\bullet\bullet_{next}}$ to the irreducible variety $W_\circ \cap W_{\bullet\bullet_{next}} \subset Q$. Thus $D_Q = W_{\bullet_{next}} \cap W_\circ \subset W_{\bullet\bullet_{next}} \cap W_\circ$.

Let S be a set of good quadrilaterals with none weakly southeast of another. Define a subvariety D_S of $W_{\bullet_{next}} \cap W_\circ$ as follows. Let T_S be the open subvariety of the the pullback of $OBS(Q_\circ)_S$ to T, on which $\dim(V_a \cap M_{d_E})$ is constant. Let D_S be the closure in D_Q of the pullback of T_S to $D_Q \subset W_\circ \cap W_{\bullet\bullet_{next}}$. T is irreducible and fibers over general points of $T_S \subset T$

are irreducible and equidimensional (equidimensional because $\dim(V_a \cap M_{d_E})$ is constant on T_S). So the pullback of T_S to D_Q is irreducible, implying that D_S is irreducible.

Let S run over all subsets of good quads with none weakly southeast of another. Let Z be an irreducible component of D_Q (not to be confused with Z used in earlier proofs). We will show that there is a set S such that $Z = D_S$.

Theorem 4.5.1. *The irreducible components of D_Q are a subset of the set of D_S where S is some set of good quadrilaterals with none weakly southeast of another.*

Proof. Here we will use d_E and a' instead of $d_E - 1$ and a''.

Case (i) $\inf(\mathbf{a}, \mathbf{a}') = \mathbf{a}$.

Then $W_{\bullet_{next}} \cap W_\circ = W_{\bullet_{next}}$ and $W_{\bullet_{next}} = D_\emptyset$ since there are no good quadrilaterals.

Case (ii) $\inf(\mathbf{a}, \mathbf{a}') \neq \mathbf{a}$

Z is an irreducible component of D_Q, so Z has the same dimension as D_Q and D_Q is a divisor of $W_{\bullet\bullet_{next}} \cap W_\circ$ so

$$\operatorname{codim}_Q Z = \operatorname{codim}_Q(W_{\bullet\bullet_{next}} \cap W_\circ) + 1$$

Let $Z_{OBS(Q_\circ)}, Z_M, Z_F$ be the image of Z in $OBS(Q_\circ), OBS(Q_\circ) \times \{M.\}$, and $T \subset OBS(Q_\circ) \times \{M.\} \times \{F_{\leq c}\}$ respectively. Let $\boxed{\ell_1} = \operatorname{codim}_{OBS(Q_\circ)} Z_{OBS(Q_\circ)}$ and let S be the set of (at most ℓ_1) quadrilaterals corresponding to the smallest closed stratum of $OBS(Q_\circ)$ in which $Z_{OBS(Q_\circ)}$ is contained. Let $\boxed{\ell_4}$ be the codimension of Z_M in $\pi_M^{-1}(Z_M)$ and for a general point in Z, let

$$\boxed{\ell_2} = \dim(V_{\sup(a,a')} \cap M_{d_E}) - \dim(V_{a'})$$

Using lemma 3.3.1, let $R = d_E, j = \dim(V_{a'})$, and $B = Z_{OBS(Q_\circ)}$. Then $\ell_4 \geq \ell_2$. Note: $R = a_j$ since there is a white checker in row $R = d_E$ and so $a_j = d_E = R$. Thus we get equality ($\ell_4 = \ell_2$) only if $\ell_2 = 0$.

Let $\boxed{\ell_5}$ be the codimension of Z_F in $\pi_F^{-1}(Z_F)$. So we get as before,

$$\operatorname{codim}_T Z_F = \ell_1 + \ell_4 + \ell_5$$

For a general point $t = (V., M., F_{\leq c}) \in Z_F$, consider the set $\{F_{c+1}\}$ where

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic

3. $V_{m(F_{c+1})} \subset F_{c+1} \subset V^{\perp}_{\underline{m(F_{c+1})}}$

4. $F_{c+1} \subset (F_c^{\perp} \cap M_{d_E})^{\perp}$

We will call this space ξ_t. For general $t \in Z_F$, the dimension of ξ_t is

$$\begin{aligned}
\dim \xi_t &= \dim(F_c^{\perp} \cap V_a^{\perp} \cap (F_c^{\perp} \cap M_{d_E})^{\perp}) - \dim(F_c) - 1 - 1 \\
&= \dim(V_a^{\perp} \cap (F_c^{\perp} \cap M_{d_E})^{\perp}) - c - 2 \\
&= \dim((V_a + (F_c^{\perp} \cap M_{d_E}))^{\perp}) - c - 2 \\
&= 2n + 1 - \dim(V_a) - \dim(F_c^{\perp} \cap M_{d_E}) + \dim(V_a \cap F_c^{\perp} \cap M_{d_E}) - c - 2 \\
&= 2n + 1 - \dim(V_a) - \dim(F_c^{\perp} \cap M_{d_E}) + \dim(V_a \cap M_{d_E}) - c - 2
\end{aligned}$$

In the first line of the above calculation, the first -1 is for choosing a line in the space given. The second -1 is for the condition that the line must be isotropic. Here, $V_a^{\perp} \cap (F_c^{\perp} \cap M_{d_E})^{\perp}$ is not isotropic by corollary 3.2.1 so this is a nontrivial condition.

We now calculate the codimension of ξ_t in Q_t. Note that $\operatorname{expcod}(W_\circ \cap W_{\bullet\bullet_{next}}) = \operatorname{codim}_Q(W_\circ \cap W_{\bullet\bullet_{next}})$ was shown in section 4.4.1.

$$\begin{aligned}
\operatorname{codim}_{Q_t} \xi_t &= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim Q_t - \dim \xi_t - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + (2n + 1 - \dim(V_{\inf(a,a'')}) - c - 2) \\
&\quad - (2n + 1 - \dim(V_a) - \dim(F_c^{\perp} \cap M_{d_E}) + \dim(V_a \cap M_{d_E}) - c - 2) \\
&\quad - (\dim(V_a) + \dim(F_c^{\perp} \cap M_{d_E - 1}) - 2\dim(V_{\inf})) \\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim \inf + \dim(F_c^{\perp} \cap M_{d_E}) \\
&\quad - \dim(F_c^{\perp} \cap M_{d_E - 1}) - \dim(V_a \cap M_{d_E}) \\
&= \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + \dim \inf + 1 - \dim(V_a \cap M_{d_E}).
\end{aligned}$$

In the last line of the calculation above, $\dim(F_c^{\perp} \cap M_{d_E}) = \dim(F_c^{\perp} \cap M_{d_E - 1}) + 1$.

Let

$$\boxed{\ell_3} = \dim(V_a \cap M_{d_E}) - \dim(V_{\inf}).$$

Then $\ell_3 \geq 0$ because $V_{\inf} \subset V_a \cap M_{d_E}$. This step contributes a codimension of $\boxed{1 - \ell_3}$ compared to $\operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}})$.

Let $\boxed{\ell_6}$ be the codimension of the fiber $p_F^{-1}(t) \subset Z \to Z_F$ in ξ_t. Picture a diagram similar to Diagram (4.4). Then $\operatorname{codim}_{Q_t}(p_F^{-1}(t)) = \operatorname{codim}_{Q_t} \xi_t + \ell_6$. So we have

$$\begin{aligned}
1 &= \operatorname{codim} Z - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= (\ell_1 + \ell_4 + \ell_5 + \operatorname{codim} \xi_t + \ell_6) - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= \ell_1 + \ell_4 + \ell_5 + (\operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) + 1 - \ell_3) + \ell_6 - \operatorname{codim}(W_\circ \cap W_{\bullet\bullet_{next}}) \\
&= 1 + \ell_1 + \ell_4 + \ell_5 - \ell_3 + \ell_6 \\
&\geq 1 + \ell_1 + \ell_2 - \ell_3.
\end{aligned}$$

The final inequality is because $\ell_4 \geq \ell_2$ and $\ell_5, \ell_6 \geq 0$. We will now show that $\ell_1 + \ell_2 - \ell_3 \geq 0$.

Label vertex $\mathbf{m}$ of $Q_\circ$ with $\dim(V_m \cap M_{d_E})$. We will compute the content of the region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. Each internal diagonal edge contributes the label of its larger vertex minus the label of its smaller vertex, a non-negative contribution. The defining corners contribute their labels (positive for $\mathbf{a}, \mathbf{a}'$ and negative for $\inf(\mathbf{a}, \mathbf{a}')$ and $\sup(\mathbf{a}, \mathbf{a}')$). Thus the total content is

$$\begin{aligned}
TC &= (\text{internal diagonal contribution}) + \dim(V_a \cap M_{d_E}) + \dim(V_{a'} \cap M_{d_E}) \\
&\quad - \dim(V_{\inf} \cap M_{d_E}) - \dim(V_{\sup} \cap M_{d_E}) \\
&\geq (\dim(V_{a'}) - \dim(V_{\sup} \cap M_{d_E})) + (\dim(V_a \cap M_{d_E}) - \dim(V_{\inf})) \\
&= -\ell_2 + \ell_3.
\end{aligned}$$

Total content is bounded above by $|S|$ which is bounded above by ℓ_1. This implies $\ell_1 \geq |S| \geq -\ell_2 + \ell_3$ and so $\ell_1 + \ell_2 - \ell_3 \geq 0$, which gives us

$$1 \geq 1 + \ell_1 + \ell_2 - \ell_3 \geq 1 + 0 = 1$$

So $\ell_1 + \ell_2 - \ell_3 = 0$. Thus equality holds in all inequalities above. In particular, $\ell_5 = \ell_6 = 0$ and $\ell_2 = \ell_4 = 0$ and $\ell_1 = \ell_3$. Note that ℓ_1 and ℓ_3 are not necessarily zero. And $Z_{OBS(Q_\circ)}$ is the stratum corresponding to S. And so all quadrilaterals have content zero except for ℓ_1 quads with content 1 in region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. We will consider two cases here:

Case $\mathbf{b} \neq \mathbf{a}$

The reader may wish to refer to Figure 4.5 for an example. Let $\mathbf{b}'' \in Q_\circ$ be the vertex of the other end of the northernmost diagonal edge emanating southeast from $\mathbf{b}$. By equality above, the internal diagonal contribution is zero, so $\mathbf{b}$ and $\mathbf{b}''$ have the same label. Applying lemma 3.4.1(b)(i) to the region below edge $\mathbf{b}\mathbf{b}''$, we have that all vertices below $\mathbf{b}\mathbf{b}''$ have the same label as well. In particular, the labels of $\mathbf{b}$ and $\mathbf{a}$ are the same. Let E be the set of edges due south of $\mathbf{b}''$ union the edge $\mathbf{bb}''$. By midsort conjecture 1, there are no white checkers in the region directly east of E, so this region is a grid of quadrilaterals. Using lemma 3.4.1(b)(ii), we get that the labels on $\mathbf{b}'$ and $\sup(\mathbf{a}, \mathbf{a}')$ are the same. Thus, we do not add content by this new region east of E, and so the total content of the region of good quads, $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}'\mathbf{b}'\mathbf{b}$, is the same as the content of the region $\inf(\mathbf{a}, \mathbf{a}')\mathbf{a}' \sup(\mathbf{a}, \mathbf{a}')\mathbf{a}$. This content is ℓ_1. Thus the ℓ_1 positive-content quadrilaterals S are a subset of the good quads.

Case $\mathbf{b} = \mathbf{a}$

The result that the ℓ_1 positive-content quads S are a subset of the good quads is immediate.

We now show that no element of S is weakly southeast of another. This portion of the proof is exactly section 5.11 in [22]. We include the paragraph here for completeness.

Fix a positive-content quadrilateral. Then its northeast, southeast, and southwest vertices have the same label. Thus by repeated application of lemma 3.4.1(b)(i), all vertices south of its southern edge are labeled the same, and there are no positive-content quadrilaterals (elements of S) south of this edge. Let E' be the union of edges due south of the northeast vertex of our positive-content quadrilateral. Repeated applications of lemma 3.4.1(b)(ii) imply that any two vertices east of E' in the same column have the same label, and there are no positive content quadrilaterals here either.

Thus $Z = D_S$ for the S described above and we've shown that the irreducible components of D_Q are a subset of $\{D_S\}_S$. □

There is no white checker in row d_E, but there is a white checker in row $d_W \geq n + 2$

This case remains to be proven.

4.5.2 s_i move with $d_E \geq n + 2$

This case remains to be proven.

4.5.3 s_0 move

This case is almost exactly the same as the case for an s_i move with a white checker in row $d_E < n$. Replace references to row d_E with row n. Otherwise the proof is the same and theorem 4.5.1 holds for nontrivial s_0 moves.

4.6 Contraction of all but one or two divisors by π

We show in this section that all divisors but possibly $D_\emptyset$ and $D_{\text{NW good quad}}$ are contracted by π. Part (a) of the theorem shows that all other D_S are contracted by π. Part (b) shows that $D_\emptyset$ is contracted by π when predicted.

We will state and prove the theorem separately for each case, however, the general strategy is as follows. In all parts, we will construct for a general point $(V, M., F.) \in \pi(D_S)$ a positive dimensional family in D_S which collapses to $(V, M., F.)$. This will prove that D_S is contracted by π to a component of codimension greater than one in $\mathrm{Cl}_{\mathrm{OGr(n,2n+1)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$, hence does not contribute to D.

4.6.1 s_0 move

Theorem 4.6.1. *(a) If $S \neq \emptyset$ and $S \neq \{$northwest good quad$\}$ then D_S is contracted by π.*

(b) If $S = \emptyset$ and the white checker in row n is in the descending checker's square $(n, \underline{c})$ and there is a white checker in a column $n + 2 \leq col \leq \underline{c+1}$ then D_S is contracted by π.

Proof of Part (a). Given a general point of D_S, $((V_m)_{\mathbf{m}\in Q_\circ}, M., F.) \in D_S$, we will produce a one-parameter family $((V'_m)_{\mathbf{m}\in Q_\circ}, M., F.)$ through $(V_m)_{\mathbf{m}\in Q_\circ}$ in the stratum $OBS(Q_\circ)_S$, fixing those V_m on the northeast border of $OBS(Q_\circ)$ and those V_m where $\mathbf{a} \prec \mathbf{m}$ along the southwest border and any $\mathbf{m}$ along the southwest border in checker board columns $1, \ldots, c$. Note: For $1 \leq i \leq 2n + 1$ we have $V_{m(M_i)} \subset M_i$ since $V_{m(M_i)}$ is on the northeast border so is fixed. Also note that for $1 \leq j \leq c + 1$ and $\underline{c+1} \leq j \leq 2n + 1$ we have $V_{m(F_j)} \subset F_j \subset V^{\perp}_{m(F_{\underline{j}})}$. These two comments hold for any element $((V'_m)_{\mathbf{m}\in Q_\circ}, M., F.)$ in the family we will describe. Note also that $V_{max(Q_\circ)}$ is fixed so this one-parameter family in D_S will be contracted by π.

Here is a description of the one-parameter family: The description is exactly as in the proof of proposition 5.13(a) in [22]. We reiterate the proof here for the purpose of checking details.

Choose a quadrilateral **stuv** in S. Name the elements of $Q_\circ$ as in Figure 4.11. $\mathbf{g_m}$ is the white checker in the column containing s. $\mathbf{f_{m-1}}$ is the next white checker to the west of $\mathbf{g_m}$. A few comments:

1. $\mathbf{g_m}$ is not necessarily a vertex within the "good quad" region; it may be north of the region.

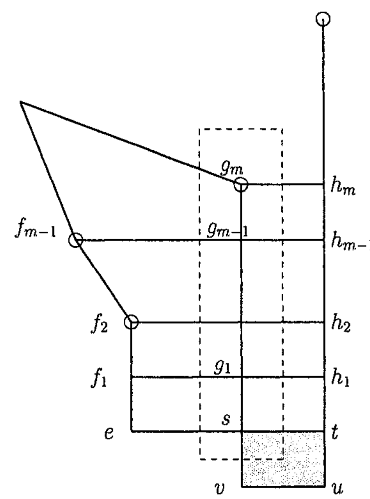


Figure 4.11: Let quadrilateral *stuv* be an element of S. Label the elements of $Q_\circ$ as in this figure.

2. The $\mathbf{s}, \mathbf{g_1}, \ldots, \mathbf{g_m}$ column is never a subset of the northeast border because there is always a column east of $\mathbf{s}, \mathbf{g_1}, \ldots, \mathbf{g_m}$ and the white checker in that column is in a more northern row than $\mathbf{g_m}$ by midsort conjecture 1.

3. If **stuv** is the northwest good quad then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$. So if $\inf(\mathbf{a}, \mathbf{a}'') = \mathbf{a}''$ then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$ is on the northeast border and is required to be fixed. And if $\inf(\mathbf{a}, \mathbf{a}'') \neq \mathbf{a}, \mathbf{a}''$ then $\mathbf{s} = \mathbf{g_m} = \inf(\mathbf{a}, \mathbf{a}'')$ has a third southeastern edge pointing due east (toward $\mathbf{a}''$).

So when **stuv** is *not* the northwest good quad, then $\mathbf{g_m}$ always has exactly one edge pointing northwest and two edges pointing southeast.

We define our family as follows: Let $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{s}, \mathbf{g_1}, \cdots, \mathbf{g_m}$. Then choose V'_s from the open set of $\mathbb{P}(V_v/V_e) \cong \mathbb{P}^1$ such that $\dim(V'_{g_i}) = \dim(\mathbf{g_i})$ for $1 \leq i \leq m$ and V'_{g_i} is defined as $V'_{g_i} = V'_s \cap V_{h_i}$. We do not get the full $\mathbb{P}^1$ of choices here because we must choose V'_s so its intersection with the "$\mathbf{h_i}$ column" gives spaces with the expected codimensions. We double check that V'_{g_i} is valid. Note that $V'_{f_i} = V_{f_i}$ is contained in V'_{g_i} since $V'_{g_i} = V'_s \cap V_{h_i}$, $V_{f_i} = V'_{f_i} = V_e \cap V_{h_i}$, and $V_e \subset V'_s$ so our containments $V'_{f_i} \subset V'_{g_i}$ make sense. $\square$

Proof of Part (b). We now suppose there is a white checker in the descending black checker position, $(n, \underline{c})$. Consider Figure 4.12. Call the white checker in position $(n, \underline{c})$, $\mathbf{d}$. Let $\mathbf{t}$ be the

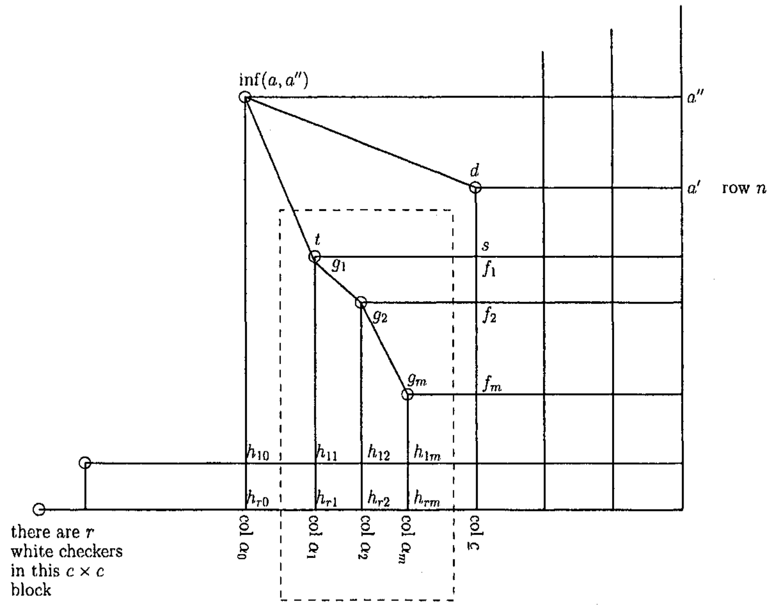


Figure 4.12: Diagram for $S = \emptyset$ and there is a white checker in row n in the descending black checker's square.

northwestern-most white checker in columns $n+2 \le col \le \underline{c+1}$. $\mathbf{g_1}, \ldots, \mathbf{g_m}$ are the m white checkers in columns $n+2 \le col \le \underline{c+1}$. These white checkers are in columns $n+2 \le \alpha_1 < \alpha_2 < \cdots < \alpha_m \le \underline{c+1}$. Let r_{α_i} be the row of the white checker called g_i. By midsort conjecture 2, $n+2 < r_{\alpha_1} < r_{\alpha_2} < \cdots < r_{\alpha_m} \le 2n+1$. $\mathbf{h_{ij}} \in Q_\circ$ is in column α_j and row of the i^{th} white checker in the southwest $c \times c$ block. There are r such white checkers in this block. We know that $V_d \neq V_t$ since $S = \emptyset$.

In $\bullet_{next}$, the black checker configuration tells us that

$$F_{\underline{c}} \cap M_n = F_{\underline{c+1}} \cap M_n = F_{\underline{c+2}} \cap M_n = \cdots = F_{n+2} \cap M_n$$

In particular,

$$F_{\underline{c}} \cap M_n = F_{\alpha_m} \cap M_n = \cdots = F_{\alpha_1} \cap M_n$$

where a_i is the column of the i^{th} white checker in the region weakly south of the critical diagonal.

We are given $((V_m), M_., F_.)$ such that

1. $(M_., F_.) \in X_{\bullet_{next}}$

2. $(V_m)_{m \in Q_\circ} \in OBS(Q_\circ)_\emptyset$

3. $V_m \subset M_{\text{row of } m} \cap F_{\text{col of } m}$ for all $\mathbf{m} \in Q_\circ$

Some other observations:

1. $S = \emptyset$ so we may assume that $V_m \neq V_{m'}$ for $\mathbf{m}, \mathbf{m}'$ opposite corners of any quadrilateral of $Q_\circ$.

2. $V_d \subset M_n \cap F_{\underline{c}} = M_n \cap F_{\alpha_1}$.

3. $V_t \subset F_{\alpha_1}$

4. $V_s = \langle V_d, V_t \rangle$ so $V_s \subset F_{\alpha_1}$. Note also that $V_s = \langle V_d, V_{g_1} \rangle$ and $V_{f_j} = \langle V_d, V_{g_j} \rangle$ with $V_{g_j} \subset F_{\alpha_j}$ and $V_d \subset M_n \cap F_{\underline{c}} = M_n \cap F_{\alpha_j}$ so $V_{f_j} \subset F_{\alpha_j}$ for $1 \leq j \leq m$.

We will describe an m-dimensional family $(V'_w)_{\mathbf{w} \in Q_\circ}$ through $(V_w)_{\mathbf{w} \in Q_\circ}$ in the stratum $OBS(Q_\circ)_\emptyset$ preserving all spaces on the northeast border and all spaces on the southwest border in columns $1 \leq col < n+1$ and in columns $\underline{c} \leq col \leq 2n+1$. In particular, we fix V_{max}.

Let $V'_w = V_w$ for $\mathbf{w} \neq \mathbf{g_1}, \ldots, \mathbf{g_m}$ and $\mathbf{w} \neq \mathbf{h_{ij}}$ for $1 \leq i \leq r$ and $1 \leq j \leq m$. Choose V'_{g_1} from the open subset of $\mathbb{P}(V_s / V_{\inf(a,a')})$ such that $V'_{g_1} \neq V_d$ and $V'_{h_{i1}} = \langle V_{h_{i0}}, V'_{g_1} \rangle$ has $\dim h_{i1}$ for $1 \leq i \leq r$. Some notes: Since $V_s \subset F_{\alpha_1}$, we know that $V'_{g_1} \subset F_{\alpha_1}$ also. And V'_{g_1} has the correct $M.$ row containment because V_s is in the same row. And $V'_{h_{i1}} = \langle V_{h_{i0}}, V'_{g_1} \rangle$ has the correct $M.$ and $F.$ containment because of $V_{h_{i0}}$ and V'_{g_1} have the correct containments. In particular, $V'_{h_{r1}} \subset F_{\alpha_1}$ because $V'_{g_1} \subset F_{\alpha_1}$ and $V_{h_{r0}} \subset F_{n+1} \subset F_{\alpha_1}$.

Now, for $2 \leq j \leq m$, choose V'_{g_j} from the open subset of $\mathbb{P}(V_{f_j} / V'_{g_{j-1}})$ such that $V'_{g_j} \neq V_{f_{j-1}}$ and $V'_{h_{ij}} = \langle V'_{g_j}, V'_{h_{i,j-1}} \rangle$ has $\dim h_{ij}$ for $1 \leq i \leq r$. Note that $V'_{g_j} \subset V_{f_j}$ and $V_{f_j} \subset F_{\alpha_j}$ so $V'_{g_j} \subset F_{\alpha_j}$. And $V'_{h_{rj}} = \langle V'_{h_{r,j-1}}, V'_{g_j} \rangle \subset F_{\alpha_j}$ because $V'_{h_{r,j-1}} \subset F_{\alpha_{j-1}} \subset F_{\alpha_j}$ and $V'_{g_j} \subset F_{\alpha_j}$.

Since the original point $((V_w)_{w \in Q_\circ}, M., F.)$ is in this family, it's nonempty. So we've described an m-dimensional ($m \geq 1$) family in D_S that collapses when we apply π. So D_S is contracted by π. □

4.6.2 s_i move with $d_E < n$

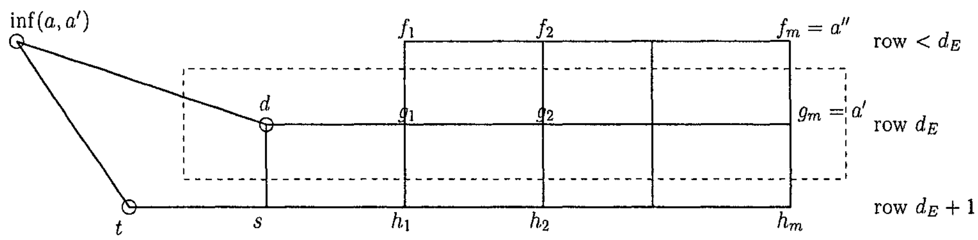


Figure 4.13: Diagram for the case $S = \emptyset$ and there is a white checker in the rising black checker square (row $d_E + 1$).

There is a white checker in row d_E

Theorem 4.6.2. *(a) If $S \neq \emptyset, \{northwest\ good\ quad\}$ then D_S is contracted by π.*

(b) (i) If $S = \emptyset$ and there is a white checker in the rising black checker square (in row d_E+1), then D_S is contracted by π.

(ii) If there is a white checker in the eastern descending black checker square (position $(d_E, \underline{c})$) and there is at least one white checker in columns $d_E + 1 \leq col \leq \underline{c+1}$ then D_S is contracted by π.

Proof of Part (a). This is exactly as the s_0-case. □

Proof of Part (b)(i). There is a white checker in row $d_E + 1$ in the rising black checker position. Name the elements of $Q_\circ$ as in Figure 4.13. Here t is the white checker in row $d_E + 1$ and d is the white checker in row d_E. We will describe a one-parameter family $(V'_m)_{\mathbf{m} \in Q_\circ}$ in $OBS(Q_\circ)$ that preserves all spaces on the northeast and southwest borders, except $V_{a'}$ (note that V_{max} is preserved). In order to show $((V'_m)_{\mathbf{m} \in Q_\circ}, M_., F_.) \in D_{S=\emptyset}$ for all elements of the family, we will need to verify that for all $V'_{a'}$ in the family, we have $V'_{a'} \subset M_{d_E}$.

Let $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{d}, \mathbf{g}_1, \ldots, \mathbf{g}_m$. Now choose V'_d from the open set of $\mathbb{P}(V_s/V_{\inf(a,a')}) \cong \mathbb{P}^1$ such that $\dim(V'_{g_i}) = \dim g_i$ where $V'_{g_i} = \langle V'_d, V_{f_i} \rangle$ for $1 \leq i \leq m$. In particular, $V'_{g_m} = \langle V'_d, V_{f_m} \rangle = \langle V'_d, V_{a''} \rangle$.

Now, $V_{a''} \subset M_{d_E - 1} \subset M_{d_E}$, and $V'_d \subset V_s$ with $V_s = \langle V_t, V_d \rangle$. We show here that $V_s \subset M_{d_E}$ which will give $V'_d \subset M_{d_E}$. $V_d \subset M_{d_E}$ and $V_t \subset M_{d_E+1} \cap F_{\underline{c+1}}$ (this is where the hypothesis that t is in row $d_E + 1$ is used). Now we have the containment $M_{d_E} \cap F_{\underline{c+1}} \subset M_{d_E+1} \cap F_{\underline{c+1}}$. By the $\bullet_{next}$-configuration, we have $\dim(M_{d_E} \cap F_{\underline{c+1}}) = \dim(M_{d_E+1} \cap F_{\underline{c+1}})$ so the two spaces are equal. This gives us $V_t \subset M_{d_E} \cap \underline{F_{c+1}} \subset M_{d_E}$. So $V_s = \langle V_t, V_d \rangle \subset M_{d_E}$. Finally, we have $V'_{g_m} = \langle V'_d, V_{f_m} \rangle \subset M_{d_E}$.

We've described a one-dimensional family in D_S which collapses to a general point in $\pi(D_S)$, so D_S is contracted by π. $\square$

Proof of Part $(b)(ii)$. Here there is a white checker in the descending black checker position in row d_E. Rows d_E and $d_W + 1$ are mirror image rows. Since there is a white checker in row d_E, there is not white checker in row $d_W + 1$, the row of the rising black checker that corresponds to the western descending black checker. With this in mind, this proof is almost identical to the s_0 move proof, with a few small, obvious changes. Mainly, F_n is replaced with F_{d_E} and references to column $n+2$ will be replaced with the column of the rising black checker in row $d_E + 1$. $\square$

There is no white checker in row d_E, but there is a white checker in row $d_W \geq n+2$

This case remains.

4.6.3 s_i move with $d_E \geq n+2$

This case remains.

4.7 Multiplicity 1

We now show that the D_S that are not contracted by π appear with multiplicity 1 in the Cartier divisor D_Q. We state the theorem once and prove it for individual cases.

Theorem 4.7.1. *(a) When $D_\emptyset$ is not contracted by π, the multiplicity of the Cartier divisor D_Q along the Weil divisor $D_\emptyset$ is 1.*

(b) If there are good quadrilaterals, the multiplicity of D_Q along $D_{\{NW\ good\ quad\}}$ is 1.

4.7.1 s_i move with $d_E < n$

There is a white checker in row d_E

Proof of Part (a). Consider the open set T' of T that lies in the preimage of the dense open stratum $(S = \emptyset)$ of $OBS(Q_\circ)$, and where $V_a \cap M_{d_E} = V_x = V_{\text{inf}}$. Let Q' be the preimage of T' in Q. Then

$$Q' \cap D_Q = Q' \cap (W_\circ \cap W_{\bullet_{next}}).$$

We want to show that $Q' \cap (W_\circ \cap W_{\bullet_{next}})$ is generically reduced. We know T' is reduced because $T' \subset T$ and T is reduced. Thus it is sufficient to show that the general fiber of $Q' \cap (W_\circ \cap W_{\bullet_{next}}) \to T'$ is reduced. Once we've shown this, then we know $W_\circ \cap W_{\bullet_{next}}$ in $W_\circ \cap W_{\bullet\bullet_{next}}$ has multiplicity one along the divisor $D_\emptyset$.

We now show that the general fiber is reduced. To build $Q' \cap (W_\circ \cap W_{\bullet_{next}})$ over a general point $(V_., M_., F_{\leq c}) \in T'$, we choose F_{c+1} such that

1. $F_c \subset F_{c+1} \subset F_c^\perp$
2. F_{c+1} is isotropic
3. $V_a \subset F_{c+1}^\perp$
4. $(M_{d_E} \cap F_c^\perp) \subset F_{c+1}^\perp$

These conditions are equivalent to choosing an isotropic F_{c+1} such that

$$F_c \subset F_{c+1} \subset [V_a + (M_{d_E} \cap F_c^\perp)]^\perp$$

Now,

$$\begin{aligned}[V_a + (M_{d_E} \cap F_c^{\perp})]^{\perp} &= [(V_a + M_{d_E}) \cap F_c^{\perp}]^{\perp} \\ &= (V_a + M_{d_E})^{\perp} + F_c \\ &= (V_a^{\perp} \cap M_{d_E}^{\perp}) + F_c\end{aligned}$$

Note that $(V_a^{\perp} \cap M_{d_E}^{\perp}) + F_c$ is actually the direct sum $(V_a^{\perp} \cap M_{d_E}^{\perp}) \oplus F_c$ because $M_{d_E}^{\perp} \cap F_c = \langle 0 \rangle$. Thus choosing F_{c+1} is equivalent to choosing a point in $O\mathbb{P}(V_a^{\perp} \cap M_{d_E}^{\perp})$. Keeping in mind that $M_{d_E} \cap V_a = V_{\text{inf}}$, we define a basis such that

- $V_{\text{inf}} = \langle e_1, \ldots, e_k \rangle$
- $V_a = \langle e_1, \ldots, e_k, e_{k+1}, \ldots, e_\ell \rangle$
- $M_{d_E} = \langle e_1, \ldots, e_k, f_1, \ldots, f_r \rangle$

where $k + r = d_E$. We choose the symmetric bilinear form B so it is standard with respect to $\langle e_1, \ldots, e_{2n+1} \rangle = \mathbb{C}^{2n+1}$. Note that $V_a \cap \langle f_1, \ldots, f_r \rangle = \langle 0 \rangle$. With this choice of basis, the rank of $O\mathbb{P}(V_a^{\perp} \cap M_{d_E}^{\perp})$ is the same as the rank of B on $\langle e_{k+1}, \ldots, e_{2n+1-\ell} \rangle \cap \langle f_1, \ldots, f_r \rangle^{\perp}$.

We consider $\mathbb{C}^{2m+1}$ where $m = n - k$. Let $V = \langle e_{k+1}, \ldots, e_\ell \rangle$ and $M = \langle f_1, \ldots, f_r \rangle$. V and M are isotropic, V is maximal in $\mathbb{C}^{2m+1}$ if and only if V_a is maximal in $\mathbb{C}^{2n+1}$, and M is maximal in $\mathbb{C}^{2m+1}$ if and only if M_{d_E} is maximal in $\mathbb{C}^{2n+1}$. Our question is now rephrased as: show $\operatorname{rank} B|_{(V^{\perp} \cap M^{\perp})} \geq 3$ for a general point. Since we are looking to find a lower bound on rank for a general point, it is sufficient to find an example of a particular V and M that yield $\operatorname{rank} B|_{(V^{\perp} \cap M^{\perp})} \geq 3$.

Example 4.7.1. We choose a new basis for V and M and a symmetric bilinear form that is standard with respect to this new basis. Let

$$M = \langle g_1, \ldots, g_{m-1} \rangle$$

and

$$V = \langle g_m + g_{m+3}, g_{m+4}, \ldots, g_{2m+1} \rangle.$$

Here we have $\dim(M) = \dim(V) = m - 1$, but this example can generalize to M and V with smaller dimensions. Then

$$M^{\perp} = \langle g_1, \ldots, g_{m-1}, g_m, g_{m+1} \rangle$$

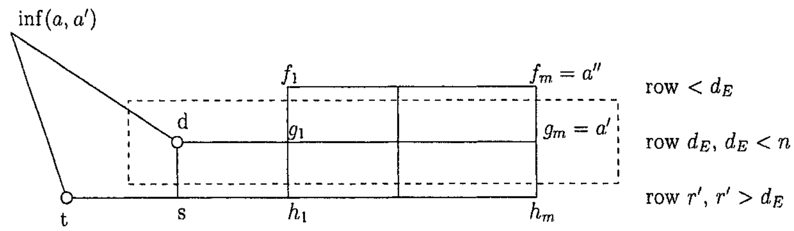


Figure 4.14:

and

$$V^{\perp} = \langle g_m, g_{m+1}, g_{m+3}, g_{m+2} - g_{m-1}; g_{m+4}, \ldots, g_{2m+1} \rangle.$$

Then the intersection is

$$M^{\perp} \cap V^{\perp} = \langle g_m, g_{m+1}, g_{m+2} - g_{m-1} \rangle$$

and rank $B|_{V^{\perp} \cap M^{\perp}} = 3$.

The example shown is for nonmaximal isotropic V and M in $\mathbb{C}^{2m+1}$ which means the example is only valid if $\dim(V_a) < n$ and $\dim(M_{d_E}) < n$. Now, $d_E < n$ by hypothesis. And if $\dim(V_a) = n$ then there are no white checkers in columns $\underline{c+1} \leq col \leq 2n+1$. In particular, there is no white checker in column $\underline{c}$. But this implies there is a white checker in column $c+1$, a contradiction to our hypothesis that column $c+1$ is devoid of white checkers. Thus this example is valid and is sufficient for showing the rank of the bilinear form is at least three for the general fiber of $Q' \cap W_{\circ} \cap W_{\bullet_{next}} \to T'$ which shows our fiber is reduced. □

Proof of Part (*b*).

We give a test family $\mathcal{F}$ through a general point $(V_{\cdot}, M_{\cdot}, F_{\cdot})$ of $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ meeting D_Q along $D_{\{\text{NW good quad}\}}$ with multiplicity 1. Label the elements of $Q_{\circ}$ as in Figure 4.14. $\mathbf{t}$ is the highest white checker in columns $d_E + 1 \leq col \leq \underline{c+1}$ and r' is the row of checker $\mathbf{t}$. We will define a family $\mathcal{F} = \{((V'_m)_{\mathbf{m} \in Q_\circ}, M'_{\cdot}, F'_{\cdot})\}$ as follows:

- $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{d}, \mathbf{g_1}, \ldots, \mathbf{g_m}$
- Choose $e_t \in V_t$ and $e_d \in V_d$ so that e_t is a generator of $V_t/V_{\inf}$ and e_d is a generator of $V_d/V_{\inf}$. Let $V'_d = \langle V_{\inf}, \mu e_t + \nu e_d \rangle$ where $[\mu, \nu] \in \mathbb{P}^1$, so V'_d varies in the pencil $\mathbb{P}(V_s/V_{\inf})$. Let $V'_{g_i} = \langle V_{f_i}, V'_d \rangle$.

- To build $M'_\cdot$, we have two possibilities: either $r' \leq d_W$ or $r' > d_W$.

  1. If $r' \leq d_W$ then let $M'_i = M_i$ for $1 \leq i < d_E$ and $r' \leq i \leq n$. And let $M'_{2n+1-i} = M_i^{\perp}$. Define $M'_{d_E} = \langle M_{d_E-1}, \mu e_t + \nu e_d \rangle$, and $M'_{dW} = (M'_{d_E})^{\perp}$. For $d_E + 1 \leq i < \min(r' - 1, n)$, choose M'_i such that

     (a) $M'_{i-1} \subset M'_i \subset (M'_{i-1})^{\perp}$

     (b) M'_i is isotropic

     (c) $V'_{m(M_i)} \subset M'_i$ (this condition is not necessary becuse there is not a white checker in row i)

     (d) $V_{m(M_i^{\perp})} \subset (M'_i)^{\perp}$

     Then define $M'_{2n+1-i} = (M'_i)^{\perp}$.

  2. If $r' > d_W$ then there are no white checkers in rows $d_E + 1 \leq row \leq n$ in columns $d_E + 1 \leq col \leq \underline{c+1}$. In the maximal case, this means at least one of the following occur:

     (a) There is a white checker in row n in a column $\underline{c} \leq col \leq 2n+1$.

     (b) There is a white checker in row $n+2$ in a column $d_E + 1 \leq col \leq \underline{c+1}$.

     (c) There is a white checker in row $n+2$ in a column $\underline{c} \leq col \leq 2n+1$.

     For the first possibility, the column of such a white checker would be less than the column of the white checker labeled **d** by midsort conjecture 1. So this white checker will serve as a blocker, causing no northwest good quadrilateral. So this possibility does not occur here. For the second possibility, we would have $n + 2 \leq d_W$. Then $r' = n+2 \leq d_W$ which goes against our hypothesis that $r' > d_W$. The third possibility is like the first: such a white checker would serve as a blocker to any checkers in columns $d_E + 1 \leq col \leq \underline{c+1}$.

     So for the maximal case, we cannot have $r' > d_W$ and still have a northwest good quadrilateral. We assume that $r' \leq d_W$ for the rest of this proof.

- Next we build $F'_\cdot$. Let $F'_j = F_j$ for $1 \leq j \leq c$ and $\underline{c} \leq j \leq 2n+1$. Now choose F'_{c+1} such that

  1. $F_c \subset F'_{c+1} \subset F_c^{\perp}$

2. F'_{c+1} is isotropic

3. $V_a \subset (F'_{c+1})^\perp$

4. $V_{m(F_{c+1})} \subset F'_{c+1}$

5. $(M_{d_E-1} \cap F_c^\perp) \subset (F'_{c+1})^\perp$

For $c+2 \le j \le \underline{c+2}$, define

$$F'_j = F'_{c+1} + [(F'_{c+1})^\perp \cap M_{r_j}]$$

where r_j is the row of the black checker in column j.

With this construction, and $\mu = 0$, we get our original general point in $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$. So $\mathcal{F} \not\subset D_Q$. Also, when $\nu = 0$, we get $V'_d = V'_t$ so $(V'_m)_{m \in Q_\circ} \in OBS(Q_\circ)_{\{\text{NW good quad}\}}$ and thus $\mathcal{F}$ meets $D_{\{\text{NW good quad}\}}$ at $\nu = 0$. We will see that D_Q contains the divisor $\nu = 0$ with multiplicity 1, proving the result.

Keep in mind that $(F'_{c+1})^\perp$ contains $V_a = V'_a$, $V'_{m(F_{c+1})} = V_{m(F_{c+1})} \subset F'_{c+1}$, and $(F'_{c+1})^\perp$ contains $M_{d_E-1} \cap F_c^\perp$ for all points of $\mathcal{F}$.

The divisor D_Q on $\mathcal{F}$ is given by

1. $V_a \subset (F'_{c+1})^\perp$

2. $(M'_{d_E} \cap F_c^\perp) \subset (F'_{c+1})^\perp$

3. $V_{m(F_{c+1})} \subset F'_{c+1}$

These three conditions are equivalent to

$$\langle V_a, M'_{d_E} \cap F_c^\perp \rangle \subset (F'_{c+1})^\perp \subset V^\perp_{m(F_{c+1})}$$

And since we already know that $V_a \subset (F'_{c+1})^\perp$ and $(F'_{c+1})^\perp \subset V^\perp_{m(F_{c+1})}$ for all points in $\mathcal{F}$, the divisor condition on $\mathcal{F}$ is equivalent to

$$(M'_{d_E} \cap F_c^\perp) \subset (F'_{c+1})^\perp$$

Now, consider $\langle M_{d_E-1}, F_c^\perp \rangle = K^m$ (this is fixed for all points of $\mathcal{F}$). Choose a basis $e_1, \ldots, e_k$ for $F_c^\perp$ and $f_1, \ldots, f_j$ for M_{d_E-1}. Here, $k = 2n+1-c$ and $j = d_E - 1$. Let $l = \dim(M_{d_E-1} \cap F_c^\perp) = d_E - 1 - c$ and $e_i = f_i$ for $1 \le i \le l$. Then $K^m = \langle e_1, \ldots, e_k, f_{l+1}, \ldots, f_j \rangle$. Define the projection $\sigma : K^m \to F_c^\perp$ by $\sigma(e_i) = e_i$ and $\sigma(f_i) = 0$. σ vanishes on $M_{d_E-1}/(M_{d_E-1} \cap F_c^\perp)$ so $(Id - \sigma)(K^m) \subset M_{d_E-1}$. Now, $M'_{d_E} = \langle M_{d_E-1}, \mu e_t + \nu e_d\rangle$, so by our condition above, D_Q is given by

$$
\begin{aligned}
&(M'_{d_E} \cap F_c^\perp) \subset (F'_{c+1})^\perp \\
\iff &\langle M_{d_E-1}, \mu e_t + \nu e_d\rangle \cap F_c^\perp \subset (F'_{c+1})^\perp \\
\iff &\langle M_{d_E-1}, \sigma(\mu e_t + \nu e_d)\rangle \cap F_c^\perp \subset (F'_{c+1})^\perp \\
\iff &(M_{d_E-1} \cap F_c^\perp) + \langle \sigma(\mu e_t + \nu e_d)\rangle \subset (F'_{c+1})^\perp \text{ since } \sigma(\mu e_t + \nu e_d) \in F_c^\perp \\
\iff &\mu\sigma(e_t) + \nu\sigma(e_d) \in (F'_{c+1})^\perp \text{ since } (M_{d_E-1} \cap F_c^\perp) \subset (F'_{c+1})^\perp \text{ for all points in } \mathcal{F} \\
\iff &\mu e_t + \nu\sigma(e_d) \in (F'_{c+1})^\perp \\
&\quad \text{since } e_t \in F_c^\perp \text{ since } \mathbf{t} \text{ is in a column less than } \underline{c} \text{ and } \sigma \text{ is the identity on } F_c^\perp \\
\iff &\nu\sigma(e_d) \in (F'_{c+1})^\perp \\
&\quad \text{since } \mathbf{t} \prec \mathbf{a} \text{ so } e_t \in V_a \text{ and } V_a \subset (F'_{c+1})^\perp
\end{aligned}
$$

Since the final statement is only true if we are in the divisor D_Q, we know this condition is not satisfied by all elements of $\mathcal{F}$ (as $\mathcal{F} \not\subset D_Q$). This tells us that $\sigma(e_d) \notin \langle V_a, M_{d_E-1} \cap F_c^\perp\rangle$ because if it were, then $\sigma(e_d) \in (F'_{c+1})^\perp$.

Thus the restriction of D_Q to $\mathcal{F}$ has two components, each with multiplicity 1. They are:

1. the hyperplane section $\{(F'_{c+1})^\perp \mid \sigma(e_d) \in (F'_{c+1})^\perp\} \subset \mathbb{P}(F_c^\perp/\langle V_a, M_{d_E-1} \cap F_c^\perp\rangle)^*$. This verifies that the multiplicity of D_Q along $D_\emptyset$ is 1 (in the special case where there is a northwest good quadrilateral).

2. The fiber for $\nu = 0$ is also a component, appearing with multiplicity 1 as desired.

□

There is no white checker in row d_E, but there is a white checker in row $d_W \geq n+2$

This case remains.

4.7.2 s_i move with $d_E \geq n+2$

This case remains.

4.7.3 s_0 move

Proof of Part (a).

Consider a general point $(V_\cdot, M_\cdot, F_\cdot) \in \mathrm{Cl}_{\mathrm{OBS}(Q_\circ)\times(X_\bullet \cup X_{\bullet_{\mathrm{next}}})} X_{\circ\bullet}$. The reader may wish to refer to Figures 4.9 and 4.10 as examples. Without loss of generality, let $F_\cdot$ be the standard flag where $F_j = \langle e_1, \ldots, e_j \rangle$. and $M_i = \langle e_{c_1}, \ldots, e_{c_i} \rangle$ where c_k is the column of the black checker in row k of the $\bullet$-configuration. Since we are considering a general point of $\mathrm{Cl}_{\mathrm{OBS}(Q_\circ)\times(X_\bullet \cup X_{\bullet_{\mathrm{next}}})} X_{\circ\bullet}$, $V \cap M_i \cap F_j = V_m$ for $\mathbf{m} \in Q_\circ$ in position (i,j) on the checker board. In particular, $\mathbf{d}$ is the white checker in position (n, c_d) and

$$V_d = \langle V_{\inf}, w + e_{\underline{c}} \rangle$$

where

$$w = \sum_{i=1+\text{ col of inf}}^{n} a_i e_i + \sum_{i=\underline{c}+1}^{c_d} b_i e_i.$$

By $\mathbf{d}$'s position, (n, c_d), we can assume that the coefficient in front of $e_{\underline{c}}$ is 1 and $b_{c_d} \neq 0$. By hypothesis, we are looking at $D_\emptyset$ and are only interested in $D_\emptyset$ when it is not contracted by π. So by theorem 4.6.1, we may assume $c_d > \underline{c}$.

We give a test family $\mathcal{F}$ through the general point $(V_\cdot, M_\cdot, F_\cdot) \in \mathrm{Cl}_{\mathrm{OBS}(Q_\circ)\times(X_\bullet \cup X_{\bullet_{\mathrm{next}}})} X_{\circ\bullet}$ meeting D_Q along $D_\emptyset$ with multiplicity 1. The family $\mathcal{F} = \{(V'_\cdot, M'_\cdot, F'_\cdot)\}$ is given by

- Fix $F'_\cdot = F_\cdot$.
- Let $M'_i = M_i$ and $(M'_i)^\perp = M_i^\perp$ for $1 \leq i \leq n-1$.
- Define $M'_n = M_{n-1} + \langle \frac{1}{2}s^2 e_{c+1} + st e_{n+1} - t^2 e_{\underline{c}} \rangle$ for $[s,t] \in \mathbb{P}^1$. Then $M'_{n+1} = (M'_n)^\perp$.
- For $\mathbf{m} \in Q_\circ$ where $\mathbf{d} \not\prec \mathbf{m}$, let $V'_m = V_m$.
- Define
  $$V'_d = V_{\inf} + \langle w + \frac{1}{2}s^2 e_{c+1} + st e_{n+1} - t^2 e_{\underline{c}} \rangle.$$
  Note that V'_d is completely determined by the choice of $[s,t]$ for M'_n.
- For $\mathbf{m} \in Q_\circ$ with $\mathbf{d} \prec \mathbf{m}$ and $\mathbf{m} \neq \mathbf{d}$, inductively define V'_m as $V'_{NE} + V'_{SW}$, the span of the vector spaces associated to the northeast and southwest corners of the quadrilateral where $\mathbf{m}$ is the southeast corner.

The set of all such $(V'_\cdot, M'_\cdot, F'_\cdot)$ is the 1-dimensional family $\mathcal{F}$.

For a point $(V'_\cdot, M'_\cdot, F'_\cdot) \in \mathcal{F}$, consider a quadrilateral in $Q_\circ$, with northeast corner at position (r_{NE}, c_{NE}) and associated vector space V_{NE} and with southwest corner at position (r_{SW}, c_{SW}) and associated vector space V_{SW}. V_{NE} has a vector $v = \sum_{i=1}^{2n+1} a_i e_i$ where $a_{c_{NE}} \neq 0$. If V_{NE} is in column c_d, then $a_{c_{NE}} = b_{c_d} \neq 0$ because $\langle w + (\frac{1}{2}s^2 e_{c+1} + ste_{n+1} - t^2 e_{\underline{c}})\rangle \subset V_{NE}$. If V_{NE} is in a column not equal to c_d, then V_{NE} inherits the nonzero coefficient $a_{c_{NE}}$ from the general point $(V_\cdot, M_\cdot, F_\cdot)$. Now, because $V_{SW} \subset F_{c_{SW}} = \langle e_1, \cdots, e_{c_{SW}}\rangle$ and $c_{SW} < c_{NE}$, all vectors of V_{SW} are of the form $w = \sum_{i=1}^{2n+1} b_i e_i$ where $b_{c_{NE}} = 0$. So $V_{NE} \neq V_{SW}$ for any quad in $Q_\circ$. Thus $V'_\cdot \in OBS(Q_\circ)_\emptyset$ for every element of $\mathcal{F}$.

When $[s,t] = [0,1]$, $(V'_\cdot, M'_\cdot, F'_\cdot)$ is the original general point $(V_\cdot, M_\cdot, F_\cdot)$. So $\mathcal{F} \not\subset D_\emptyset$. When $[s,t] = [1,0]$, $M'_n = M_{n-1} + \langle e_{c+1}\rangle$ which implies $(M'_n \cap F_c^\perp) \subset F_{c+1}^\perp$, which makes $(M'_\cdot, F'_\cdot) \in X_{\bullet_{next}}$. So $\mathcal{F}$ meets $D_\emptyset$.

Now, moving away from the original general point, let $s = 1$. Then we have $M'_n = M'_{n-1} + \langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\rangle$. A point in $\mathcal{F}$ is in $D_\emptyset$ if and only if

$$\begin{aligned} & M'_n \cap (F'_c)^\perp \subset (F'_{c+1})^\perp \\ \iff & \dim(M'_n \cap F_{c+1}) \geq 1 \quad \text{by lemma 4.3.1} \\ \iff & \dim((M'_{n-1} + \langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\rangle) \cap F_{c+1}) \geq 1. \end{aligned}$$

Now, for $(M'_\cdot, F'_\cdot) \in X_\bullet \cup X_{\bullet_{next}}$, we have that $\dim(M'_{n-1} \cap F_{c+1}) = 0$, so

$$\dim((M'_{n-1} + \langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\rangle) \cap F_{c+1}) \geq 1 \iff \langle \frac{1}{2}e_{c+1} + te_{n+1} - t^2 e_{\underline{c}}\rangle \subset F_{c+1}.$$

This is true if and only if $-t^2 = 0$ and $t = 0$ (since $n+1 > c+1$ and $\underline{c} > c+1$), a multiplicity 1 condition. So $\mathcal{F}$ meets D_Q with multiplicity 1 at a point of $D_\emptyset$, and therefore D_Q has multiplicity one along $D_\emptyset$. $\square$

Proof of Part (b).

We give a test family $\mathcal{F}$ through a general point $(V_\cdot, M_\cdot, F_\cdot)$ of $\mathrm{Cl}_{\mathrm{OBS}(Q_\circ)\times(X_\bullet \cup X_{\bullet_{\mathrm{next}}})} X_{\circ\bullet}$ meeting D_Q along $D_{\{\text{NW good quad}\}}$ with multiplicity 1. Label the elements of $Q_\circ$ as in Figure 4.15. See Figure 4.10 as an example.

- **t** is the highest white checker in columns $n+2 \leq col \leq \underline{c+1}$.

- r is the row of checker **t**, $\underline{r} = 2n+1-r < n$.

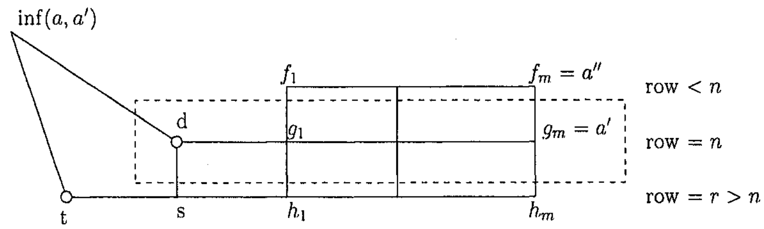


Figure 4.15:

The family $\mathcal{F} = \{(V'_., M'_., F'_.)\}$ is given by

- $V'_m = V_m$ for $\mathbf{m} \neq \mathbf{d}, \mathbf{g_1}, \ldots, \mathbf{g_m}$

- Choose $e_t \in V_t$ and $e_d \in V_d$ so that e_t is a generator of V_t/V_{inf} and e_d is a generator of V_d/V_{inf}. Let $V'_d = \langle V_{\mathrm{inf}}, \mu e_t + \nu e_d \rangle$ where $[\mu, \nu] \in \mathbb{P}^1$ and $V'_{g_i} = \langle V_{f_i}, V'_d \rangle$ has $\dim(V'_{g_i}) = \dim(\mathbf{g_i})$. So V'_d varies in an open subset of $\mathbb{P}(V_s/V_{\mathrm{inf}})$. Let $V'_{g_i} = \langle V_{f_i}, V'_d \rangle$.

- For $1 \leq i \leq \underline{r}$, let $M'_i = M_i$ and $(M'_i)^{\perp} = M_i^{\perp}$.

- We make the following observation: there is no white checker in row $\underline{r}+1$ because $\underline{r}+1 = 2n+1-r+1 = 2n+2-r$ and row r has checker $\mathbf{t}$ in it. By maximality, since there are no white checkers in rows $n+2 \leq row \leq r-1$, we must have white checkers in rows $\underline{r}+2 \leq row \leq n$.

- Choose a line L such that

  1. $L \not\subset M_{\underline{r}}$
  2. $L \subset M_r = M_{\underline{r}}^{\perp}$
  3. L is isotropic
  4. $L \subset ((V'_d)^{\perp} \cap V_{f_m}^{\perp}) = (V_{f_m}^{\perp} \cap \langle \mu e_t + \nu e_d \rangle^{\perp})$
  5. $L \subset F_c^{\perp}$

  Then define

  $$M'_{\underline{r}+1} = M_{\underline{r}} + L$$

  This is a valid choice for $M'_{\underline{r}+1}$ if the following are true:

  1. $M_{\underline{r}} \subset M'_{\underline{r}+1} \subset M_r$

2. $\dim(M'_{\underline{r}+1}) = \underline{r}+1$

3. $M'_{\underline{r}+1}$ is isotropic

4. $V'_{m(M_{\underline{r}+1})} \subset M'_{\underline{r}+1}$

5. $V'_{m(M_{r-1})} \subset (M'_{\underline{r}+1})^{\perp} = M'_{r-1}$

We will show that the above are all satisfied by our choice of $M'_{\underline{r}+1}$. $M_{\underline{r}} \subset M_{\underline{r}} + L = M'_{\underline{r}+1}$ and both $M_{\underline{r}}$ and L are contained in M_r so 1 is satisfied. $L \not\subset M_{\underline{r}}$ so $\dim(M_{\underline{r}} + L) = \underline{r}+1$, proving 2. L is isotropic and contained in $M_{\underline{r}}^{\perp}$ so $M_{\underline{r}} + L$ is isotropic which is 3. There is no white checker in row $\underline{r}$ so 4 is not a new condition. There are also no white checkers in rows $n+1, \ldots, r-1$ so

$$V'_{m(M_{r-1})} = V'_{m(M_n)} = V'_{g_m} = \langle \mu e_t + \nu e_d \rangle + V_{f_m}.$$

Now, $\langle \mu e_t + \nu e_d \rangle + V_{f_m} \subset V_s \subset M_r$ which implies $M_{\underline{r}} \subset (\langle \mu e_t + \nu e_d \rangle + V_{f_m})^{\perp}$. And $L \subset (\langle \mu e_t + \nu e_d \rangle + V_{f_m})^{\perp}$ by hypothesis. So

$$\begin{aligned} & M_{\underline{r}} + L = M_{\underline{r}+1} \subset (\langle \mu e_t + \nu e_d \rangle + V_{f_m})^{\perp} \\ \iff & \langle \mu e_t + \nu e_d \rangle + V_{f_m} \subset M_{\underline{r}+1}^{\perp} \\ \iff & V'_{m(M_{r-1})} \subset M_{\underline{r}+1}^{\perp} \end{aligned}$$

Thus showing 5.

- Now define for $\underline{r}+2 \leq i \leq n$
$$M'_i = M'_{\underline{r}+1} + V'_{m(M_i)}.$$
Then with perps, we have $M'_{\cdot}$.

- We now build the $F'_{\cdot}$ part of the family. Let $F'_j = F_j$ for $1 \leq j \leq c$. Choose F'_{c+1} such that

1. $F_c \subset F'_{c+1} \subset F_c^{\perp}$

2. F'_{c+1} is isotropic

3. $V'_a \subset (F'_{c+1})^{\perp}$

4. $V'_{m(F_{c+1})} \subset F'_{c+1}$ (There is no white checker in column $c+1$, so this is not a new condition.)

5. $(M'_{n-1} \cap F_c^{\perp}) \subset (F'_{c+1})^{\perp}$

Consider condition 5. $M'_{n-1} \cap F_c^\perp = (M'_{\underline{r}+1} + V_{fm}) \cap F_c^\perp = (M_{\underline{r}} + L + V_{fm}) \cap F_c^\perp$. So

$$\begin{aligned} &(M'_{n-1} \cap F_c^\perp) \subset (F'_{c+1})^\perp \\ \Longleftrightarrow\ & F'_{c+1} \subset ((M_{\underline{r}} + L + V_{f_m}) \cap F_c^\perp)^\perp \\ \Longleftrightarrow\ & F'_{c+1} \subset (M_{\underline{r}} + L + V_{f_m})^\perp + F_c \\ \Longleftrightarrow\ & F'_{c+1} \subset (M_r \cap L^\perp \cap F_{f_m}^\perp) + F_c \end{aligned}$$

Note that $L \subset F_c^\perp$ so $F_c \subset L^\perp$. So we have

$$F'_{c+1} \subset (M_r \cap L^\perp \cap V_{f_m}^\perp) + F_c \subset (L^\perp + F_c) = L^\perp$$

which implies $L \subset F'_{\underline{c+1}}$.

- Now define F'_j for $c+2 \leq j \leq n$ as

$$F'_j = F'_{c+1} + (F'_{\underline{c+1}} \cap M'_{r_j})$$

where r_j is the row of the black checker in column j of the $\bullet$-configuration. With perps, this gives $F'_\cdot$.

With $\mu = 0$ we get the original point $(V_\cdot, M_\cdot, F_\cdot)$, so $\mathcal{F} \not\subset D_Q$. With $\nu = 0$, $V'_d = V'_t$ so $(V'_\cdot, M'_\cdot, F'_\cdot) \in D_{\text{NW good quad}}$ and $\mathcal{F}$ meets $D_{\text{NW good quad}}$. We will see that D_Q contains the divisor $\nu = 0$ with multiplicity 1, proving the result.

The divisor D_Q on $\mathcal{F}$ is given by

1. $V_a \subset (F'_{c+1})^\perp$
2. $V_{m(F_{c+1})} \subset F'_{c+1}$
3. $(M'_n \cap F_c^\perp) \subset F'_{\underline{c+1}}$

Note that $(M'_{n-1} \cap F_c^\perp) \subset F'_{\underline{c+1}}$ along with conditions 1 and 2 are satisfied by all points of $\mathcal{F}$. A point of $\mathcal{F}$ is in D_Q

$$\begin{aligned} \Longleftrightarrow\ & (M'_n \cap F_c^\perp) \subset F'_{\underline{c+1}} \\ \Longleftrightarrow\ & ((M_{\underline{r}+1} + V'_{g_m}) \cap F_c^\perp) \subset F'_{\underline{c+1}} \\ \Longleftrightarrow\ & ((M_{\underline{r}} + L + V_{f_m} + \langle \mu e_t + \nu e_d \rangle) \cap F_c^\perp) \subset F'_{\underline{c+1}} \\ \Longleftrightarrow\ & (M_{\underline{r}} + V_{f_m} + \langle \mu e_t + \nu e_d \rangle) \cap F_c^\perp + L \subset F'_{\underline{c+1}} \\ \Longleftrightarrow\ & [(M_{\underline{r}} + V_{f_m}) + \langle \mu e_t + \nu e_d \rangle] \cap F_c^\perp \subset F'_{\underline{c+1}} \end{aligned}$$

The last equivalence is because $L \subset F'_{\underline{c+1}}$.

Let $\ell = \dim((M_{\underline{r}} + V_{f_m}) \cap F_c^{\perp})$. Choose a basis $e_1, \ldots, e_k$ for $F_c^{\perp}$ and $f_1, \ldots, f_j$ for $M_{\underline{r}} + V_{f_m}$ such that $e_i = f_i$ for $1 \leq i \leq \ell$. Then

$$(M_{\underline{r}} + V_{f_m}) + F_c^{\perp} = \langle e_1, \ldots, e_k, f_{\ell+1}, \ldots, f_j \rangle.$$

Define a projection

$$\sigma : (M_{\underline{r}} + V_{f_m}) + F_c^{\perp} \to F_c^{\perp}$$

by $\sigma(e_i) = e_i$ for $1 \leq i \leq k$ and $\sigma(f_i) = 0$ for $\ell + 1 \leq i \leq j$. The projection σ vanishes on $(M_{\underline{r}} + V_{f_m})/[(M_{\underline{r}} + V_{f_m}) \cap F_c^{\perp}]$ so $(Id - \sigma)((M_{\underline{r}} + V_{f_m}) + F_c^{\perp}) \subset M_{\underline{r}} + V_{f_m}$. Note that σ is fixed for all points in the family $\mathcal{F}$. So we can continue the equivalence:

$$\begin{aligned}
& [(M_{\underline{r}} + V_{f_m}) + \langle \mu e_t + \nu e_d \rangle] \cap F_c^{\perp} \subset F'_{\underline{c+1}} \\
\iff & [(M_{\underline{r}} + V_{f_m}) + \langle \sigma(\mu e_t + \nu e_d) \rangle] \cap F_c^{\perp} \subset F'_{\underline{c+1}} \\
\iff & (M_{\underline{r}} + V_{f_m}) \cap F_c^{\perp} + \langle \sigma(\mu e_t + \nu e_d) \rangle \subset F'_{\underline{c+1}} \quad \text{since } \sigma(\mu e_t + \nu e_d) \in F_c^{\perp} \\
\iff & \sigma(\mu e_t + \nu e_d) \in F'_{\underline{c+1}} \quad \text{since } M_{\underline{r}} + V_{f_m} \subset M'_{n-1} \text{ and } M'_{n-1} \cap F_c^{\perp} \subset F'_{\underline{c+1}}\ \forall \text{ pts of } \mathcal{F} \\
\iff & \mu\sigma(e_t) + \nu\sigma(e_d) \in F'_{\underline{c+1}} \\
\iff & \mu e_t + \nu\sigma(e_d) \in F'_{\underline{c+1}} \quad \text{since } \sigma \text{ is the identity on } F_c^{\perp} \\
\iff & \nu\sigma(e_d) \in F'_{\underline{c+1}} \quad \text{since } t \prec a, \text{ so } \langle e_t \rangle \subset V_a \subset F'_{\underline{c+1}} \quad \forall \text{ pts of } \mathcal{F}
\end{aligned}$$

The final statement is true only if we are in the divisor D_Q. Since $\mathcal{F} \not\subset D_Q$, this statement is not satisfied by all points in $\mathcal{F}$.

Thus the restriction of D_Q to $\mathcal{F}$ has two components, each with multiplicity 1. They are:

1. the hyperplane section $\{F'_{c+1} \mid \sigma(e_d) \in F'_{\underline{c+1}}\} \subset \mathcal{F}$
2. the fiber for $\nu = 0$ is also a component, appearing with multiplicity 1 as desired.

$\square$

4.8 Connecting divisors to white checker moves

For the cases of nontrivial s_0 moves and s_i moves with a white checker in row $d_E < n$, we have two loose ends to tie up to conclude the proof of the type B_n geometric Littlewood-Richardson rule. These loose ends are exactly section 5.16 of [22]. We state them and include the proofs nearly verbatim.

The loose ends:

1. $\pi(D_\emptyset) = \overline{X}_{\circ_{stay}\bullet_{next}}$ and / or $\pi(D_{\text{NW good quad}}) = \overline{X}_{\circ_{swap}\bullet_{next}}$.

2. Furthermore $\overline{X}_{\circ_{stay}\bullet_{next}}$ appears with multiplicity 1 in $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$ if $D_\emptyset$ appears with multiplicity 1 in $\mathrm{Cl}_{\mathrm{OBS(Q_\circ)\times(X_\bullet \cup X_{\bullet_{next}})}} \mathrm{X}_{\circ\bullet}$, and similarly for $\overline{X}_{\circ_{swap}\bullet_{next}}$ and $D_{\text{NW good quad}}$.

Both are a consequence of the next result ((2) using the fact that π is birational).

Theorem 4.8.1. *The morphism π induces birational maps from*

(a) $D_\emptyset$ to $\overline{X}_{\circ_{stay}\bullet_{next}}$ and

(b) $D_{NW\ good\ quad}$ to $\overline{X}_{\circ_{swap}\bullet_{next}}$.

Proof. (a) The inverse rational map $\overline{X}_{\circ_{stay}\bullet_{next}} \dashrightarrow D_\emptyset$ is given by the morphism $X_{\circ_{stay}\bullet_{next}} \to OBS(Q_\circ) \times X_{\bullet_{next}}$: by definition $X_{\circ_{stay}\bullet_{next}}$ parameterizes isotropic flags $M_.$ and $F_.$ in $\bullet_{next}$-position, as well as the maximal isotropic space V and isotropic spaces $V \cap M_i \cap F_j$, which correspond to elements of $Q_\circ$ (and $\dim(V \cap M_i \cap F_j)$ equals the corresponding element of $Q_\circ$).

(b) The inverse rational map $\overline{X}_{\circ_{swap}\bullet_{next}} \dashrightarrow D_{\text{NW good quad}}$ is similarly given by the morphism $X_{\circ_{swap}\bullet_{next}} \to OBS(Q_\circ) \times X_{\bullet_{next}}$, by way of the locally closed immersion $OBS(Q_{\circ_{swap}})_\emptyset \cong OBS(Q_\circ)_{\text{NW good quad}} \hookrightarrow OBS(Q_\circ)$.

$\square$

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