DISSERTATION

PROPERTIES OF THE RECONSTRUCTION ALGORITHM AND ASSOCIATED SCATTERING TRANSFORM FOR ADMITTIVITIES IN THE PLANE

Submitted by

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In partial fulfillment of the requirements

for the degree of Doctor of Philosophy

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ABSTRACT OF DISSERTATION

PROPERTIES OF THE RECONSTRUCTION ALGORITHM AND ASSOCIATED SCATTERING TRANSFORM FOR ADMITTIVITIES IN THE PLANE

We consider the inverse admittivity problem in dimension two. The focus of this dissertation is to develop some properties of the scattering transform $S_\gamma(k)$ with $\gamma \in W^{1,p}(\Omega)$ and to develop properties of the exponentially growing solutions to the admittivity equation. We consider the case when the potential matrix is Hermitian and the definition of the potential matrix used by Francini [Inverse Problems, 16, 2000]. These exponentially growing solutions play a role in developing a reconstruction algorithm from the Dirichlet-to-Neumann map of γ. A boundary integral equation is derived relating the Dirichlet-to-Neumann map of γ to the exponentially growing solutions to the admittivity equation.

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Chapter 1

INTRODUCTION

A very interesting problem in the theory of partial differential equations is the inverse conductivity problem. It has a rich history, a lot of beautiful mathematics, and many practical appplications.

Let $\Omega \subset \mathbf{R}^2$ be a bounded, simply connected Lipschitz domain. Let $\gamma(x) = \sigma(x) + i\omega\epsilon(x)$, where σ and ϵ are real measurable functions defined in Ω with both σ and ϵ bounded away from zero and infinity, and ω a small positive number.

The forward (electrical) admittivity problem is to determine $u \in H^1(\Omega)$ given γ of the Dirichlet problem

$$\begin{aligned} \nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega) \end{aligned} \tag{1.1}$$

Here γ is the admittivity, σ is conductivity and ϵ is permittivity. In chapter 3, we will briefly discuss this terminology. We will call $\nabla \cdot \gamma \nabla u = 0$ the admittivity equation and if $\gamma = \sigma$, the conductivity equation.

This boundary value problem is elliptic, and describes the behavior of the electric potential in a conductive body Ω.

Define the Dirichlet-to-Neumann map for γ as

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega), \\ \langle \Lambda_\gamma f, g \rangle &= \int_\Omega \gamma \nabla u \cdot \nabla \bar{v} \end{aligned} \tag{1.2}$$

where $v \in H^1(\Omega)$ with trace g on the boundary. This operator is compact and self-adjoint.

Physically, f represents a voltage distribution on the boundary of a physical body Ω with an unknown electrical admittivity distribution γ. Let u be a function that represents the electric potential inside the body that solves the admittivity equation (1.1) and agrees with f on the boundary. The inverse admittivity problem is to determine an unknown admittivity distribution inside a physical body from electrical measurements on the surface of the body.

We will address the inverse admittivity problem (called the inverse conductivity problem if there is no permittivity) of determining γ from knowledge of Λ_γ. Historically, much more research has been done on the inverse conductivity problem, there are only two known theoretical papers on the inverse admittivity problem [Fra00] and [Buk07].

The Dirichlet-to-Neumann map (D-N map) Λ_γ can be interpreted as the voltage-to-current density map

$$\begin{aligned} \Lambda_\gamma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ \Lambda_\gamma : f &\mapsto \gamma \frac{\partial u}{\partial \nu}|_{\delta\Omega}, \end{aligned} \tag{1.3}$$

which depends nonlinearly on the admittivity γ.

Using integration by parts (Green's Theorem), we see that

$$\begin{aligned}\langle \Lambda_\gamma f, g\rangle &= \int_\Omega \gamma \nabla u \cdot \nabla \bar{v} \\ &= \int_{\partial\Omega} (\gamma \frac{\partial u}{\partial \nu})\bar{v} - \int_\Omega (\nabla \cdot (\gamma \nabla u))\bar{v} \\ &= \int_{\partial\Omega} (\gamma \frac{\partial u}{\partial \nu})\bar{v}\end{aligned} \tag{1.4}$$

It is now easy to see why we called the map Dirichlet-to-Neumann.

The inverse (electrical) conductivity problem of Calderón [Cal80] is to uniquely determine σ from the knowledge of the Dirichlet-to-Neumann map Λ_σ. That is, does $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ imply $\sigma_1 = \sigma_2$?. The conductivity σ may have some additional regularity assumptions. The inverse conductivity problem was solved in numerous function spaces, we will give a brief history of the problem in chapter 4. The inverse admittivity problem was recently solved by Francini [Fra00] in $W^{1,p}(\Omega)$.

There are several important questions related to the Dirichlet-to-Neumann map Λ_γ:

- The injectivity of Λ_γ (uniqueness),
- The continuity of Λ_γ^{-1} (stability),
- Determination of the range of Λ_γ^{-1} (existence),
- The reconstruction of γ from Λ_γ,
- Numerical implementation of the reconstruction algorithm.

The uniqueness question is extremely important since an affirmative answer tells us that two different conductivities can be distinguished from boundary measurements. Stability is important for the analysis of the propagation of errors in the data and also for numerical algorithms. The existence question is answered in the constructive uniqueness proofs. The reconstruction question is important both from a theoretical and applied point of view. Of course, the numerical implementation of the reconstruction algorithm is important from a practical point of view. Note that these issues we mentioned are not necessarily independent. As we will see in chapter three, the uniqueness proof can provide a reconstruction algorithm.

Perhaps the most important application of the inverse admittivity (conductivity) problem is the medical imaging technique known as electrical impedance tomography (EIT). EIT allows us to produce a picture of the electrical conductivity distribution inside a person. An important application of the two dimensional geometry considered here is cross-sectional chest imaging. By applying current patterns on electrodes around the perimeter of a person's chest and measuring the resulting voltage on the electrodes, the solution to the inverse conductivity problem will give us a two dimensional image of a cross-section of the chest. This is doable since the tissues and organs in the body have different conductivities. Some applications of EIT in this case include monitoring of heart and lung function, diagnosis of pulmonary embolus, diagnosis of pulmonary edema, and monitoring for internal bleeding. EIT has been used in clinical trials. Other applications of EIT include underground contaminant detection, materials testing and nondestructive testing. There are many papers that discuss EIT, see for example, [CIN99], [Bor02] or [Hol93].

Of course, the electrical conductivity and permittivity vary between tissues and organs in the body, the frequency of the applied current, as well as depending on temperature and physiological factors. The fact that the electrical conductivity and permittivity vary is also what makes EIT useful and allows us to form an image. Most algorithms just reconstruct conductivity. However, physiological features can be visible in the permittivity image that are not visible in the conductivity image. Dr. David Isaacson gave a talk at the AIP meeting 2007 in Vancouver and presented data that illustrates this fact. Also there is a paper that shows reconstructed images of conductivity and permittivity changes of banana and cucumber [KOW+07] that demonstrate the dependence of the reconstructed conductivity and permittivity on frequency and also display the above property.

Distinguishing between certain lung pathologies (example: pneumothorax vs lung distension) is a promising application of imaging permittivity.

The inverse admittivity problem is a very difficult problem in the theory of PDEs since it is nonlinear and ill-posed. The reconstruction problem is ill-posed in the sense that large changes in the conductivity can correspond to small changes in the boundary data. The Dirichlet-to-Neumann map is a pseudo-differential operator, whose kernel can be computed from its Fourier symbol [LU89]. Furthermore, the problem is more difficult in dimension two than dimension n ≥ 3. The main reason is because the Schwartz kernel of Λ_γ is a distribution of 2(n-1) variables, but the conductivity σ throughout the domain, is a function of n variables. Thus, in dimension two the problem is formally determined, while in dimension n ≥ 3 is overdetermined.

The reconstruction of the admittivity γ by the method studied here involves an important object called the scattering transform (to be defined below). The scattering transform is a very useful tool to use in the inverse conductivity problem since the scattering transform can be computed from the boundary measurements, and the conductivity can be computed from the scattering transform.

Scattering theory for a certain first order elliptic system (to be defined below) also has important consequences for non-linear evolution equations [Sun94a], [Sun94b], [Sun94c]. These works show that a scattering transform for the elliptic system considered here can be used to transform a nonlinear system of PDEs in 2 dimensions called the Davey-Stewartson II system into a linear evolution equation. Thus, the scattering transform is an important entity arising in inverse scattering theory, EIT, and evolution equations, and a study of its properties will be a main (but not sole) focus of this dissertation.

As we will see, when the conductivity $\sigma \in W^{2,p}(\Omega)$, the scattering transform $\mathbf{t}(k)$ plays a major role in solving the inverse conductivity problem. In this context, Nachman [Nac96], Siltanen [Sil99], Mueller and Siltanen [MS03], Knudsen [Knu02], Knudsen, Lassas, Mueller and Siltanen [KLMS07] have developed some properties for the scattering transform $\mathbf{t}$.

Brown and Uhlmann's argument used to solve the inverse conductivity problem for $\sigma \in W^{1,p}(\Omega)$ [BU97] relies on scattering theory that had been developed earlier by Beals and Coifman [BC88] and Sung [Sun94a], [Sun94b], [Sun94c].

Kundsen [Knu02] and Brown [Bro01] established some properties for the scattering transform for a real measurable function $\sigma \in W^{1,p}(\Omega)$. Francini [Fra00] also used some properties for a real measurable function $\sigma \in W^{1,p}(\Omega)$ that was used in the proof of the uniqueness problem in the admittivity case, such properties were established by Sung [Sun94a]

We will give a brief overview of previous work done in establishing some properties of the two-dimensional scattering transform $S_\sigma(k)$ used in the first order elliptic system in the real case.

Francini's work [Fra00] provided the first proof that the Dirichlet-to-Neumann map uniquely determines the admittivity γ and established some properties of the electric potential u and the scattering transform. However [Fra00] does not contain a complete set of formulas that constitute a direct reconstruction alogorithm, allowing one to go from the Dirichlet-to-Neumann data to γ, as, for example, [Nac96] and [AP06] do in the real case. This dissertation provides some of the required equations, although some may be improved upon in the future.

In this dissertation, we derived several new properties of the scattering transform corresponding to a complex measurable function $\gamma \in W^{1,p}(\Omega)$. We have also established some properties of the exponentially growing solutions to the admittivity equation (1.1) that will be useful for a reconstruction algorithm of the admittivity from the Dirichlet-to-Neumann map. We also prove that a modified definition of the potential matrix Q_γ from that of [Fra00] leads to symmetry properties of the entries of the exponentially growing solution matrix and scattering transform.

The dissertation is organized as follows. In chapter 2, we will briefly discuss the mathematical model of our problem. In chapter 3 we will give a brief history and discuss the uniqueness of the admittivity (conductivity) in some important Sobolev spaces. Chapter 4 is a discussion on known properties for the scattering transforms corresponding to conductivity in several different Sobolev spaces. Chapters 5 and 6 contain the most important contributions to this dissertation. In these chapters we will prove new properties of the associated scattering transform for admittivities in the plane, and also some identities involving the exponentially growing solutions to the admittivity equation that will be useful for a future reconstruction algorithm for admittivities in the Sobolev space $W^{1,p}(\Omega)$ in the plane. Francini defines a certain potential matrix that was useful to prove new properties of the exponentially growing solutions to the admittivity equation, but the definition wasn't useful in establishing properties of the off-diagonal entries of the scattering transform. We define a new potential matrix that turns out to be useful in establishing symmetry properties of the off-diagonal entries of the scattering transform, but we can't use it to establish any properties of the exponentially growing solutions to the admittivity equation. In Appendix A we will recall some important function spaces, notations, and notations used in the inverse admittivity conductivity problem.

Chapter 2

A MATHEMATICAL MODEL

We will now briefly discuss some terminology and also how the forward electrical admittivity (conductivity) problem evolved.

Electrical impedance describes the measure of an object's opposition to a sinusoidal alternating current (AC).

Electrical impedance extends the concept of resistance in AC circuits by not only describing the relative amplitudes of voltages and currents, but also the changes in their relative phases.

We define **Magnetic permeability** as the degree of magnetization of a material that responds linearly to an applied magnetic field.

Materials with high permeabilties allow magnetic flux through more easily than others.

Permittivity is the measure of the ability of a material to store a change.

The **Electrical conductivity** measures the ease with which a steady current can flow.

The **electrical admittivity** (EIT) is a complex-valued function $\gamma = \sigma + i\omega\epsilon$ consisting of the electrical conductivity (the real part) along with the electrical permittivity (imaginary part), where ω is the time frequency of the AC current.

Electrical impedance tomography is a medical imaging system that produces a picture of electrical conductivity distribution insides a human body.

Let's briefly discuss the mathematical model for EIT [Fra00]. This will motivates why we are looking at the Dirichlet problem and the Dirichlet-to-Neumann map.

Let's denote, respectively, E and H the electric and magnetic fields. We first need Maxwell's equations for electromagnetic waves of frequency ω :

$$curlE = -i\omega\mu H \tag{2.1}$$

$$curlH = \sigma E + i\omega\epsilon E, \tag{2.2}$$

where σ, ϵ, and μ are, respectively,(electrical) conductivity, (electrical) electrical permittivity and magnetic permeability of the medium and $i = \sqrt{-1}$.

It has been shown that the magnetic permability of the human body is a very small number and its dependence on time is negligible. Since we know that the coefficient μ is sufficiently small inside the human body, there exists a potential function u such that

$$E = -\nabla u \tag{2.3}$$

By taking the divergence of (2.2) and using (2.3), we get

$$div(\gamma\nabla u) = 0, \tag{2.4}$$

where $\gamma = \sigma + i\omega\epsilon$ is called the admittivity of the body.

We want to perform some analysis on the boundary of the body Ω with data of interest including the voltage $u|_{\partial\Omega}$ and the current $\gamma\frac{\partial u}{\partial \nu}|_{\partial\Omega}$, where v is the unit outer normal to the boundary $\delta\Omega$.

For each $\gamma \in L^\infty(\Omega)$, and for every $f \in H^{1/2}$ we define the Dirichlet-to-Neumann map Λ_γ via

$$\langle \Lambda_\gamma f, \mathrm{g} \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v,$$

where u is the unique solution of the admittivity equation such that

$$u|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega).$$

Also, $\Lambda_\gamma f$ will have values in the dual space of $H^{1/2}(\Omega)$, $H^{-1/2}(\Omega)$.

The mathematical model to interpret EIT measurements is known as the inverse admittivity (conductivity) problem. Designing a practical algorithm for EIT is hard because the partial differential equation used in this framework is both nonlinear and ill-posed. Nonlinearity makes the mathematical solution for the inverse problem of interest very difficult, and ill-posedness means that a small error in the practical measurements may cause a large error in the reconstructed image.

Chapter 3

SOME UNIQUENESS AND RECONSTRUCTION RESULTS IN TWO DIMENSIONS

The inverse conductivity problem has a rich history in dimension two. Most results are for the inverse conductivity problem (neglecting permittivity).

The inverse conductivity problem was first introduced by A.P. Calderón in 1980 [Cal80]. He gave some preliminary results for this inverse problem with conductivities close to a constant. This works is in dimension $n \geq 2$. He really faced this problem while working as an engineer in Argentina in the 1950's.

Nachman in [Nac96] solved the inverse conductivity problem for $\sigma \in W^{2,p}(\Omega)$, $p > 1$. Nachman's work involved conductivities having two weak derivatives. An important feature of Nachman's proof is that it is constructive; it outlines a direct method for solving σ without using iterative techniques such as least squares minimization, linearization, or layer-stripping. He uses the d-bar method of inverse scattering to prove uniqueness and to give a reconstruction algorithm. The reconstruction algorithm has been numerically implemented by Mueller and Siltanen [SMI00] and [MS03] and other works have followed. There is also a recent paper [IMNS04] with experimental data.

Brown and Uhlmann in [BU97] only needed to work with conductivities having only one derivative and gave a positive answer to the uniqueness question for conductivities in $W^{1,p}(\Omega)$, $p > 2$. Brown and Uhlmann's argument relied on scattering theory that had been developed by Ablowitz and Fokas [AF84], Beals and Coifman [BC88], and Sung [Sun94a], [Sun94b], [Sun94c]. Knudsen [Knu02] came up with useful properties of the scattering transform and exponentially growing solutions to the conductivity equations. Knudsen and Tamasan in [KT04] came up with a reconstruction algorithm from the uniqueness proof in [BU97] for the two dimensional inverse conductivity problem. Knudsen [Knu03] also did some numerical work for the reconstruction algorithm.

The papers [Nac96] and [BU97] use similar ideas in solving the inverse conductivity problem; they look for special exponentially growing solutions to a certain equation, and they used a d-bar equation. However they used different approaches to get there. Nachman reduces the conductivity equation to a Schrödinger equation while Brown and Uhlmann transform the conductivity equation to a first order elliptic system.

Astala and Päivärinta [AP06] solved the inverse conductivity problem in dimension two with $\sigma \in L^\infty(\Omega)$. Their proof also uses a d-bar equation. There is also a reconstruction method from the uniqueness argument. There is no regularity condition required on the boundary.

In 1999 E. Francini solved the inverse conductivity problem for $\gamma = \sigma + i\omega\epsilon \in W^{1,p}(\Omega)$ [Fra00], where ω is small. Her work was the first major theoretical result which solved the inverse admittivity problem involving both conductivity and permittivity. However as we will see in chapter 5, there are no important and useful symmetries properties involving the off-diagonal entries scattering transform since a certain matrix potential is not Hermitian. We still get some useful properties with this approach as we will see in chapter six.

A recent paper by Bukhgeim [Buk07] claims to have global uniqueness for the Calderón problem with admittivity.

We will first briefly look at the inverse conductivity problem with $\sigma \in W^{2,p}(\Omega)$, $\sigma \in W^{1,p}(\Omega)$, and $\gamma \in W^{1,p}(\Omega)$. In all such inverse problems, the scattering transform plays an important role in the reconstruction of σ.

Next we will study the approach Brown and Uhlmann [BU97] used to solve the inverse conductivity problem with $\sigma \in W^{1,p}(\Omega)$. They approached this problem by analyzing a certain first-order elliptic system. We will also briefly look at a reconstruction algorithm by Knudsen [Knu02].

In section three of this chapter, we state Francini's approach to solve the inverse admittivity problem for a complex measurable function $\gamma = \sigma + i\omega\epsilon \in W^{1,p}(\Omega)$, the proof is very similar to that of [BU97]. We will see in chapters five and six, Francini's definition of the matrix potential has pros and cons.

We will complete this chapter by looking at another approach in which the matrix potential is defined so that it is Hermitian. This definition leads to symmetry properties with $\gamma = \sigma + i\omega\epsilon \in W^{1,p}(\Omega)$. However we could not obtain a formula connecting the associated scattering transform with the definition due to some technical complications we will discuss in chapter six. As a result, a reconstruction algorithm from that approach is incomplete.

3.1 Uniqueness and Reconstruction for $\sigma \in W^{2,p}(\Omega)$

In [Nac96], Nachman proved the following classical result:

Theorem 1. *Let Ω be a bounded, simply connected, Lipschitz domain in $\mathbf{R}^2$. Let σ_1,$\sigma_2 \in W^{2,p}(\Omega)$ for some $p > 1$, and have positive lower bounds. If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then $\sigma_1 = \sigma_2$.*

Nachman's proof was constructive. It outlines a direct method for reconstructing the conductivity from knowledge of the D-N map.

Let's briefly look at some important ideas used in for the reconstruction method [Nac96].

The general inverse conductivity problem can be reduced to the case where the conductivity $\sigma \equiv 1$ near $\partial\Omega$ in a nontrivial way by using analytic continuation to extend σ to be 1 in the neighborhood of the boundary of a larger domain Ω_2 [Nac96]. After proving this construction, Nachman assumes $\sigma = 1$ in a neighborhood of $\partial\Omega$.

Throughout this section we may assume $\sigma \in L^\infty_+(\Omega) := \{\sigma \in L^\infty(\Omega) : \sigma > 0 \text{ and } \sigma^{-1} \in L^\infty(\Omega)\}$. We do assume additional smoothness on σ. We also assume that $0 < c \leq \sigma$ for all $x \in \Omega$.

The original approach in [Nac96] was to reduce the conductivity equation $\nabla \cdot \sigma \nabla u = 0$ to the Schrödinger equation $(-\Delta + q)v = 0$ where $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. By letting $v = \sigma^{1/2}u$, one can show there is a 1-1 correspondence between solutions to the conductivity equation and solutions to the Schrödinger equation.

The approach in [Nac96] is to seek solutions of the equation

$$(-\Delta + q)\psi(x,k) = 0, \ \ \mathrm{x} \in \mathbf{R}^2 \ \ \mathrm{k} \in \mathbf{C} - \{0\} \tag{3.1}$$

where $k = k_1 + ik_2 \in \mathbf{C}$ is a complex parameter and $\psi(x,k) \sim e^{ikx}$ for large $|k|$ and large $|x|$ in a certain sense. We associate $x \in \mathbf{R}^2$ with $x_1 + ix_2$ and so this is complex multiplication in the exponent. More precisely, we want $\psi(x,k)$ to be such that

$$\mu(x,k) := e^{-ikx}\psi(x,k),\ \ x \in \mathbf{R}^2,\ k \in \mathbf{C} - \{0\} \tag{3.2}$$

with $\mu(\cdot,k) - 1 \in W^{1,p}(\mathbf{R}^2)$, where $p > 2$. From [Nac96] we know there is a unique solution ψ for every k if $q := \sigma^{-1/2}\Delta\sigma^{1/2}$. The exponentially growing solutions ψ to the Schrödinger equation are the key to the reconstruction and are known as Faddeev solutions. Such solutions were introduced by Faddeev [Fad66] in the context of mathematical physics.

The next lemma is an important result in [Nac96], the absence of exceptional points.

Lemma 2. *([Nac96]) Let $q = \sigma^{-1/2}\Delta\sigma^{1/2}$, with $\sigma \in W^{2,p}(\mathbf{R}^2)$, $1 < p < 2$. Then for any $k \in \mathbf{C} - \{0\}$ there exists $\psi(\cdot,k)$ such that*

$$(-\Delta + q)\psi(x,k) = 0,\ \ x \in \mathbf{R}^2 \tag{3.3}$$

with $w(\cdot,k) = \psi(\cdot,k)e^{-ikz} - 1 \in W^{1,\tilde{p}}(\mathbf{R}^2)$, where $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{1}{2}$.

The solution of the inverse conductivity problem is based on the scattering transform

$$\mathbf{t}(k) = \int_{\mathbf{R}^2} e_k(x)q(x)\mu(x,k)dx, \tag{3.4}$$

where $e_k(x) = e^{i(kx+\bar{k}\bar{x})}$ and the functions μ are given by (3.2) with $\mu(\cdot,k) - 1 \in W^{1,p}(R^2)$.

The transform $\mathbf{t}$ is nonlinear since in the integrand of (3.4) μ depends on q. Since μ is asymptotically near to one for large $|k|$, $\mathbf{t}(k)$ behaves similarly to the Fourier transform of q(x) evaluated at the point $(-2k_1, 2k_2) \in R^2$, where $|k|$

is large. It can be shown that smoothness in q corresponds to decay in $\mathbf{t}$ and symmetries in q imply symmetries in $\mathbf{t}$ [MS03]. These claims will be made more precise in section 4.1.

Define a single-layer operator S_k by

$$(S_k\phi) = \int_{\delta\Omega} G_k(x-y)\phi(y)d\mu(y), \tag{3.5}$$

where the kernel G_k of S_k is defined by

$$G_k := e^{ikx}g_k(x), \; -\Delta G_k = \delta, \tag{3.6}$$

$$g_k(x) := \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} \frac{e^{ix\xi}}{\xi(\bar{\xi}+2k)}d\xi, \; (-\Delta - 4ik\bar{\delta})g_k = \delta. \tag{3.7}$$

G_k is called the Faddeev Green's function.

The function μ in (3.2) is the unique solution of the Lippmann-Schwinger type equation $\mu = 1 - g_k * (q\mu)$ with $\mu - 1 \in W^{1,p}(\Omega)$, and $*$ denotes convolution.

Nachman shows the trace on $\partial\Omega$ of the function $\psi(\cdot, k)$ satisfies the integral equation [Nac96]

$$\psi(\cdot,k)|_{\partial\Omega} = e^{ikx}|_{\partial\Omega} - S_k(\Lambda_\sigma - \Lambda_1)\psi(\cdot,k), \tag{3.8}$$

$k \in \mathbf{C} - \{0\}$, Λ_σ is the voltage-to-current map when Ω contains the conductivity distribution and Λ_1 is the Dirichlet-to-Neumann map of the homogeneous conductivity 1. The equation (3.8) is a Fredholm equation of the second kind and uniquely solvable in $H^{1/2}(\partial\Omega)$ for any $k \in \mathbf{C} - \{0\}$.

We will now briefly discuss what we meant by exceptional points for the conductivity equation [Knu02]. Let $V = \left\{\mathbf{C}^2 \backslash \{0\} : k\cdot k = 0\right\}$, where $k\cdot k = k_1^2 + k_2^2$

is the real inner product. We look for exponentially growing solutions to the Schrödinger equation

$$(-\Delta + q)\psi(x,k) = 0, \ \ \mathrm{x} \in \mathbf{R}^2 \tag{3.9}$$

where $\psi(x,k) \sim e^{ikx}$ as $|x| \to \infty$.

The points $\xi \in V$ in which there is no unique exponentially growing solutions are called exceptional. As discussed in [Knu02], when the conductivity $\sigma \in W^{2,\infty}(\Omega)$ and $\sigma = 1$ in a neighborhood of $\partial\Omega$, the solvability of the boundary integral equation (3.8) is equivalent to k not being exceptional.

As mentioned in [Knu02], the boundary integral equation (3.8) makes sense for $\sigma \in L^\infty_+(\Omega)$, provided $\sigma = 1$ in a neighborhood of $\partial\Omega$. We say $k \in V$ is not an exceptional point for σ if the boundary integral equation (3.8) has a unique solution in $H^{1/2}(\Omega)$.

The following is from Theorem 2.4.10 of [Knu02]. Let $\sigma \in L^\infty_+(\Omega)$ and $\sigma = 1$ in a neighborhood of $\partial\Omega$ and $\mathbf{R}^2\backslash\Omega$. Then for k not exceptional, there is a solution $\phi(\cdot,k)$ to the conductivity equation in $\mathbf{R}^2$ with $\phi(\cdot,k)e^{-ikx} - 1 \in L^2_\delta(\mathbf{R}^2)$ where $-1 < \delta < 0$.

We define exceptional points analogously for the admittivity case.

Although the scattering transform $\mathbf{t}$ is not a physically measurable function, Nachman shows that it is related to the Dirichlet-to-Neumann map in the following integral equation

$$\mathbf{t}(k) = \int_{\partial\Omega} e_k(x)(\Lambda_\sigma - \Lambda_1)\psi(x,k)d\mu(x), \tag{3.10}$$

where $\psi(x,k)$ satisfies equation (3.1). The scattering transform $\mathbf{t}(k)$ plays a key role in recovering the conductivity from the map Λ_σ. Let's briefly discuss why this is true.

Nachman [Nac96] shows the following d-bar equation holds

$$\frac{\partial}{\partial \bar{k}} \mu(z,k) = \frac{\mathbf{t}(k)}{4\pi \bar{k}} e_{-k}(z) \overline{\mu(z,k)} \tag{3.11}$$

in a weighted Sobolev space $W^{1,p}_{-B} = \{(1+|\cdot|)^{1/2} f \in W^{1,\tilde{p}}\}$, $B > 1/\tilde{p}$, $1/\tilde{p} + 1/p = 1/2$ and (3.11) has a unique solution for all x.

Nachman shows that the unique solution of (3.11) satisfies the Fredholm integral equation of the second kind

$$\mu(x,k) = 1 + \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} \frac{\mathbf{t}(k')}{(k-k')\bar{k}'} e_{-z}(k') \overline{\mu(x,k')} dk_1' dk_2'. \tag{3.12}$$

The function μ defined in (3.2) is important since Nachman shows the conductivity σ can be reconstructed from

$$\lim_{k \to 0} \mu(x,k) = \sigma^{1/2}(x). \tag{3.13}$$

The main idea of [Nac96] is finding $\mathbf{t}$ from the boundary measurements Λ_σ and then determining σ from the knowledge of $\mathbf{t}$.

In summary, the main steps of the reconstruction algorithm for conductivities for $\sigma \in W^{2,p}(\Omega)$ are

(1) Compute the trace on $\partial\Omega$ of the function $\psi(\cdot,k)$ from the boundary data using equation (3.8).

(2) Compute $\mathbf{t}(k)$ by (3.10).

(3) Solve the $\bar{\partial}$ equation (3.11) for $\mu(z,k)$.

(4) Reconstruct σ using equation (3.13).

3.2 Uniqueness and Reconstruction for $\sigma \in W^{1,p}(\Omega)$

Nachman's approach required essentially two derivatives when he converted the conductivity equation into the Schrödinger equation. The classical 1996 paper by Nachman soon was sharpened for conductivities in $W^{1,p}(\Omega)$, $p > 2$ by Brown and Uhlmann in 1997 [BU97]. The main result from this paper is

Theorem 3. *Let Ω be a bounded, simply connected, Lipschitz domain in $\mathbf{R}^2$. Let $\sigma_1, \sigma_2 \in W^{1,p}(\Omega)$, $p > 2$, and have positive lower bounds. If $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$, then $\sigma_1 = \sigma_2$.*

Let's sketch out the main ideas used in this theorem. The key idea of the proof of Theorem 2 is the following. Instead of reducing the conductivity equation to the Schrödinger equation, it is reduced to a first order elliptic system.

Define the differential operators ∂ and $\bar{\partial}$ by

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right). \tag{3.14}$$

Let $u \in H^1(\Omega)$ be a solution of the conductivity equation (1.1) and let

$$\begin{pmatrix} v \\ w \end{pmatrix} = \sigma^{1/2} \begin{pmatrix} \partial u \\ \bar{\partial} u \end{pmatrix}. \tag{3.15}$$

It is easy to verify that

$$(D - Q_\sigma) \begin{pmatrix} v \\ w \end{pmatrix} = 0, \tag{3.16}$$

where the matrix potential Q_σ and matrix operator D are defined by

$$Q_\sigma = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} \tag{3.17}$$

with $q = -\frac{1}{2}\partial \log \sigma$. By assumption $q \in L^p(\Omega)$, $p > 2$. This allowed Brown and Uhlmann to work with only conductivities involving only one derivative since $q = -\frac{\partial \sigma^{1/2}}{\sigma^{1/2}}$.

Similar to Nachman's approach to use some scattering theory to solve the two-dimensional inverse conductivity problem with essentially two derivatives, Brown and Uhlmann [BU97] used a scattering theory for this first-order elliptic system to solve the inverse conductivity problem with $\sigma \in W^{1,p}(\Omega)$.

The scattering theory for this elliptic system was developed by Ablowitz and Fokas [AF84], and Beals and Coifmann [BC88]. This method was used by Sung [Sun94a], [Sun94b], [Sun94c] to study a nonlinear PDE. Sung has shown that a scattering transform (to be defined below) for the elliptic system above could be used to transform the Davey-Stewartson II system (a nonlinear system of PDEs in 2-D) into a linear evolution equation.

Brown and Uhlmann looked for a family of exponentially growing solutions which solved (3.16) in the whole plane. We will now briefly discuss this idea. We look for special solutions of the elliptic system

$$(D - Q_\sigma)\psi = 0 \tag{3.18}$$

of the form

$$\psi(z,k) = M(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}, \tag{3.19}$$

where M is a 2×2 matrix valued function which converges to the identity matrix I as $|z| \to \infty$. We say that ψ is a family of growing exponential solutions of (3.18). We will call M the Jost matrix.

Typically we will require $Q_\sigma \in L^p_c(\mathbf{R}^2)$ be Hermitian.

Notice ψ satisfies equation (3.18) if and only if M satisfies

$$(D_k - Q_\sigma)M = 0 \text{ in } \mathbf{R}^2 \tag{3.20}$$

where D_k is the matrix operator defined by

$$D_k A = \begin{pmatrix} \bar\partial a_{11} & (\bar\partial - ik)a_{12} \\ (\partial + ik)a_{21} & \partial a_{22} \end{pmatrix} \tag{3.21}$$

and Q_σ is the matrix potential.

The entries of (3.20) are given by

$$\bar\partial m_{11}(z,k) = q(z)m_{21}(z,k) \tag{3.22}$$

$$(\partial + ik)m_{21}(z,k) = \bar q(z)m_{11}(z,k) \tag{3.23}$$

$$(\bar\partial - ik)m_{12}(z,k) = q(z)m_{22}(z,k) \tag{3.24}$$

$$\partial m_{22}(z,k) = \bar q(z)m_{12}(z,k). \tag{3.25}$$

Some useful identities are

$$(\partial + ik)u = e_{-k}\partial(e_k u) \tag{3.26}$$

$$(\bar\partial - ik)u = e_{\bar k}\bar\partial(e_{-\bar k}u). \tag{3.27}$$

where $e_k \equiv e_k(\cdot)$.

These identities follow from simple differentiation. Let us establish (3.26),

$$\begin{aligned}
\partial(e_k u) &= u\partial(e_k) + e_k\partial u \\
&= u(\partial e^{2i(k_1x_1 - k_2x_2)})e_k + e_k\partial u \\
&= u\frac{1}{2}(\frac{\partial}{\partial x_1}(2i(k_1x_1 - k_2x_2)) - i\frac{\partial}{\partial x_2}(2i(k_1x_1 - k_2x_2)))e_k + e_k u \\
&= u(\frac{1}{2}(2ik_1 + 2i^2k_2))e_k + e_k\partial u \\
&= uike_k + e_k\partial u \\
&= e_k(\partial + ik)u.
\end{aligned} \tag{3.28}$$

The following is a key result about uniqueness of the solution $M(z,k)$ to the equation (3.20).

Theorem 4. *(Theorem 2.3 of [BU97]) Let $Q \in L^p(\boldsymbol{R}^2, M_2)$, $p > 2$, be an off-diagonal, hermitian, compactly supported matrix. Then for any $k \in \boldsymbol{C}$, the equation (3.20) has a unique solution M(z,k) with $M(z,k) - 1 \in L^r(\boldsymbol{R}^2)$, $r > 2$. Moreover, for any $r > 2p/(p-2)$ we have*

$$sup_{z\in \boldsymbol{C}} \|M(z,\cdot) - I\|_{L^r(\boldsymbol{R}^2, M_2)} < C \tag{3.29}$$

where C depends on p,r,Q.

The papers [BBR01] and [Knu02] mentioned (without proof) that due to the symmetry in D_k and Q_σ and the uniqueness of M(z,k), we have

$$m_{11}(z,k) = \overline{m_{22}(z,\bar{k})} \tag{3.30}$$

and

$$m_{21}(z,k) = \overline{m_{12}(z,\bar{k})}. \tag{3.31}$$

We will prove these relations hold for the complex case provided we extend the definition of Q_γ to be Hermitian, in chapter 6.

We will need the standard Cauchy transforms in chapter 5, denoted by ∂^{-1} and $\bar{\partial}^{-1}$

$$\bar{\partial}^{-1} f(z) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(w)}{z-w} d\mu(w) \tag{3.32}$$

and

$$\partial^{-1} f(z) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(w)}{\bar{z}-\bar{w}} d\mu(w), \tag{3.33}$$

where $d\mu$ is the Lebesgue measure in $\mathbf{R}^2$

Define the scattering transform $S_\sigma(k)$ of the potential matrix Q_σ by

$$S_\sigma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} (Q_\sigma M)^{off}(z,k)d\mu(z). \tag{3.34}$$

Here $(Q_\sigma M)^{off}$ denotes the off-diagonal part of $Q_\sigma M$.

A short calculation shows the scattering transform $S_\sigma(k)$ can be written

$$S_\sigma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} 0 & e^{-i\bar{k}z}q(z)\psi_{22}(z,k) \\ -e^{ik\bar{z}}\bar{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z) \tag{3.35}$$

The following result allows us to recover $M(z,k)$ from a d-bar equation, and also gives us a relationship between $M(z,k)$ and the scattering transform $S_\sigma(k)$.

Theorem 5. *(Theorem A (iv) of [BU97]) Assume that the hypotheses hold as in the Theorem 4. Then the scattering transform $S_\sigma(k) \in L^2(\boldsymbol{R}^2, M_2)$. Moreover, for each fixed $z \in \boldsymbol{R}^2$, the map $k \mapsto M(z,k)$ is a differentiable map from $\boldsymbol{C}$ into $L^r_\beta(\boldsymbol{R}^2, M_2)$ and we have the following d-bar equation*

$$\bar{\partial}_k M(z,k) = M(z,\bar{k})E_k(z)S_\sigma(k) \tag{3.36}$$

where $\bar{\partial}_k$ is the standard complex differential operator with respect to $\bar{k}$ and

$$E_k(z) = \begin{pmatrix} e_{\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix}$$

The map $Q_\sigma \to S_\sigma$ is called the scattering map. The scattering transform S_σ is the key object object in the solution of the inverse problem.

It was mentioned in [Knu02] that the non-physical scattering transform S_σ is related to the Fourier transform of q for large k. In chapter 6, we will show that this relationship is true in the complex case.

To solve the inverse conductivity problem for $\sigma \in W^{1,p}(\Omega)$ is to show Λ_σ determines S_σ uniquely and that S_σ determines σ uniquely.

In [BBR01], the authors established several properties of the exponentially growing solutions to the admittivity equation under certain conditions. In particular, they established a boundary integral that connects the off-diagonal entries of the scattering transform $S_\sigma(k)$ and the Dirichlet-to-Neumann map. Knudsen and Tamasan in [KT04] came up with a reconstruction algorithm for the inverse conductivity problem with $\sigma \in W^{1+\delta,p}$ with $p > 2$ and $0 < \delta < 1$, they used some results of [BBR01]. The reconstruction method uses the reduction of the conductivity equation to a first order elliptic system, and applying the d-bar method of inverse scattering theory to this elliptic system. The key idea they developed was coming up with an explicit method for the computation of the scattering transform S_σ of the potential q in terms of Λ_σ. Knudsen and Tamasan work in the Sobolev space $W^{1+\epsilon,p}$, which is slightly more restrictive than what the uniqueness proof required. The remaining ideas of this section come from mostly [Knu02], but we first need a key result from [BBR01].

We will first use the idea of [BBR01] to show that there is a relationship between the off-diagonal entries of the scattering transform $S_\sigma(k)$ and the Dirichlet-to-Neumann map using the exponentially growing solutions $u(z,k)$ to the conductivity equation. The authors generalized the formula in Theorem 7 which was first established by Alessandrini [Ale90], they also used some of the ideas of [Liu97]. We have included a sketch of the proof of this result since it plays a major role in the reconstruction algorithm of $\sigma \in W^{1,p}(\Omega)$.

We will first need the following theorem from [BBR01]. We will prove some analogous theorems for admittivities in $W^{1,p}(\mathbf{R}^2)$ in chapter six.

Theorem 6. *Suppose there is a constant $C > 0$ such that $\frac{1}{C} < \sigma_i(x) < C$ for every $x \in \Omega$. and $\sigma - 1 \in W^{1,p}(\boldsymbol{R}^2)$ is compactly supported. If $k \in \boldsymbol{C} - \{0\}$, then* *there is a unique solution $u(Q,x,k)$ to the conductivity equation such that*

$$e^{-ikx}u(Q,x,k) - \frac{1}{ik} \in W^{1,r}(\boldsymbol{R}^2) \quad \textit{for any} \quad 2 < r < \infty, \tag{3.37}$$

$$e^{-ikx}u(Q,x,k) - \frac{1}{ik} = (\partial + ik)^{-1}(\sigma^{-1/2}(z)m_{11}(Q,x,k) - 1), \tag{3.38}$$

and

$$\bar{\partial}(e^{-ikx}u(Q,x,k) - \frac{1}{ik}) = \sigma^{-1/2}(z)m_{21}(Q,x,k). \tag{3.39}$$

We will need some more results and notations before we prove an important theorem, these materials are coming from [BBR01].

Remark: We assume there is a constant $C > 0$ such that $\frac{1}{C} < \sigma_i(x) < C$ for every $x \in \Omega$. In [BBR01], they have shown that it is sufficient to look at the case when $\sigma_i = 1$, $\frac{\partial \sigma_i}{\partial \eta} = 0$ on $\partial\Omega$, $\sigma - 1 \in W^{1,p}(\mathbf{R}^2)$ is compactly supported, and $\sigma_i \in C^{1+\epsilon}(\bar{\Omega})$, $i = 1, 2$.

We let Q_i be the potential matrices associated to the conductivities σ_i,

$$Q_i = \begin{pmatrix} 0 & q_i \\ \bar{q}_i & 0 \end{pmatrix}$$

where $q_i = -\frac{1}{2}\partial \log \sigma_i$. Note that in this case since σ is real, $\bar{q}_i = -\frac{1}{2}\overline{\partial \log \sigma_i} = -\frac{1}{2}\bar{\partial} \log \sigma_i$. We let $M(Q_i,x,k)$ be the corresponding Jost matrices and let $u(Q_i,x,k)$ be the solutions to the conductivity equation as in Theorem 6.

For convenience, we will rewrite (3.38) as

$$e^{-ikx}u(Q_i,x,k) - \frac{1}{ik} = R(Q_i,x,k) \tag{3.40}$$

From (3.38) and (3.39), we get

$$\sigma_i^{1/2}\partial u(Q_i,x,k) = e^{ikx}m_{11}(Q_i,x,k), \tag{3.41}$$

$$\sigma_i^{1/2}\bar{\partial}u(Q_i,x,k) = e^{ikx}m_{21}(Q_i,x,k), \tag{3.42}$$

Now we ready to state a key theorem and sketch the proof of it [BBR01].

Theorem 7. *(Proposition 3.1 of [BBR01]) Let σ_i, $i = 1, 2$ be as the above remark and $k \in \mathbf{C}\setminus\{0\}$ Then*

$$\frac{\pi}{\bar{k}}((S_{\sigma_1})_{21}(k) - (S_{\sigma_2})_{21}(k)) = \frac{1}{2}\int_{\partial\Omega} \overline{u(Q_1,x,-k)}(\Lambda_{\sigma_1} - \Lambda_{\sigma_2})u(Q_2,x,k)d\mu(x), \tag{3.43}$$

where $(S_{\sigma_i})_{21}(k)$ and $(S_{\sigma_i})_{12}$ are the off-diagonal entries of the scattering transform and $u(Q_i, x, k)$ are the exponentially growing solutions to the conductivity equations corresponding to σ_i, $i = 1, 2$. Moreover, we have

$$(S_\sigma)_{21}(k) = \frac{-i\pi}{2}\int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_\sigma - \Lambda_{\sigma_1})u(\cdot,k)d\mu(x), \tag{3.44}$$

Proof. Let's first discuss some of the key steps to the proof.

Define

$$\begin{aligned} I = &\int_\Omega \bar{\partial}(\overline{u(Q_1,x,-k)}u(Q_2,x,k)){\sigma_2}^{1/2}\partial{\sigma_1}^{1/2}dx \\ &- \int_\Omega \partial(\overline{u(Q_1,x,-k)}u(Q_2,x,k)){\sigma_1}^{1/2}\bar{\partial}{\sigma_2}^{1/2}dx. \end{aligned} \tag{3.45}$$

We will sketch the argument in two steps. We will first show that Step 1:

$$I = \frac{1}{k}(S_{\sigma_1})_{21}(-\bar{k}) + \frac{1}{\bar{k}}(S_{\sigma_2})_{21}(k).$$

Next we will prove that Step 2:

$$-I = \frac{1}{2}\int_{\partial\Omega} \overline{u(Q_1,x,-k)}(\Lambda_{\sigma_1} - \Lambda_{\sigma_2})u(Q_2,x,k)d\mu(x).$$

Once we establish steps 1 and 2, we can conclude

$$\frac{\pi}{k}(S_{\sigma_1})_{12}(-\bar{k}) + \frac{\pi}{\bar{k}}(S_{\sigma_2})_{21}(k) = \frac{1}{2}\int_{\partial\Omega} \overline{u(Q_1,x,-k)}(\Lambda_{\sigma_2} - \Lambda_{\sigma_1})u(Q_2,x,k)d\mu(x). \tag{3.46}$$

Now set $\sigma_2 = \sigma_1$ in Ω and then

$$\frac{1}{k}(S_{\sigma_1})_{12}(-\bar{k}) = -\frac{1}{\bar{k}}(S_{\sigma_1})_{21}(k). \tag{3.47}$$

We see that (3.46) and (3.47) will get us equation (3.43).

Step 1: Let $q_i = -\frac{1}{2}\partial \log \sigma_i$ and $\bar{q}_i = -\frac{1}{2}\bar{\partial} \log \sigma_i$, $i = 1, 2$ defined as usual. Also, let us set $M(Q_i, x, k)$ be the standard Jost matrix asymptotic to I for large $|k|$ with respect to the potential matrix Q_i, $i = 1, 2$.

From (3.41),(3.42) and (3.40), we can set $I = I_1 + I_2$, where

$$\begin{aligned} I_1 = &- \frac{1}{ik}\int_\Omega e_k(x)\overline{m_{11}(Q_1, x, -k)}q_1(x){\sigma_2}^{1/2}(x)dx \\ &- \frac{1}{i\bar{k}}\int_\Omega e_k(x)m_{21}(Q_2, x, k)q_1(x){\sigma_1}^{1/2}(x)dx \\ &+ \frac{1}{ik}\int_\Omega e_k(x)\overline{m_{21}(Q_1, x, -k)}\bar{q}_2(x){\sigma_2}^{1/2}(x)dx \\ &+ \frac{1}{i\bar{k}}\int_\Omega e_x(z)m_{11}(Q_2, x, k)\bar{q}_2(x){\sigma_1}^{1/2}(x)dx \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} I_2 = &- \int_\Omega e_k(x)\overline{m_{11}(Q_1, x, -k)}q_1(x){\sigma_2}^{1/2}(x)R(Q_2, z, k)dx \\ &- \int_\Omega e_k(x)m_{21}(Q_2, x, k)q_1(x){\sigma_1}^{1/2}(x)\overline{R(Q_1, x, -k)}dx \\ &+ \int_\Omega e_k(x)\overline{m_{21}(Q_1, z, -k)}\bar{q}_2(x){\sigma_2}^{1/2}(x)R(Q_2, x, k)dx \\ &+ \int_\Omega e_k(x)m_{11}(Q_2, x, k)\bar{q}_2(x){\sigma_1}^{1/2}(x)\overline{R(Q_1, x, -k)}dx, \end{aligned} \tag{3.49}$$

where $R(Q_i, x, k)$ is as (3.40), $i = 1, 2$.

We will now show that

$$I_1 = \frac{\pi}{k}(S_{\sigma_1})_{12}(-\bar{k}) + \frac{\pi}{\bar{k}}(S_{\sigma_2})_{21}(k) \quad \textit{and} \quad I_2 = 0$$

We know that

$$\frac{\pi}{k}(S_{\sigma_1})_{12}(-\bar{k}) = -\frac{1}{ik}\int_\Omega e_k(x)\overline{m_{11}(Q_1, x, -k)}q_1(x)dx$$

and

$$\frac{\pi}{\bar{k}}(S_{\sigma_2})_{21}(k) = \frac{1}{i\bar{k}}\int_\Omega e_k(x)m_{11}(Q_2, x, k)\bar{q}_2(x)dx.$$

Thus, by (3.30)

$$\begin{aligned} I_1 = & \frac{\pi}{k}(S_{\sigma_1})_{12}(-\bar{k}) + \frac{\pi}{\bar{k}}(S_{\sigma_2})_{21}(k) \\ & + \frac{1}{ik}\Big[-\int_\Omega e_k(x)\overline{m_{11}(Q_1, x, -\bar{k})}q_1(x)({\sigma_2}^{1/2}(x) - 1)dx \\ & + \int_\Omega e_k(x)\overline{m_{21}(Q_1, x, -\bar{k})}\bar{q}_2(x){\sigma_2}^{1/2}(x)dx\Big] \\ & + \frac{1}{i\bar{k}}\Big[-\int_\Omega e_k(x)m_{21}(Q_2, x, k)q_1(x){\sigma_1}^{1/2}(x)dx \\ & + \int_\Omega e_k(x)m_{11}(Q_2, x, k)\bar{q}_2(x)({\sigma_1}^{1/2}(x) - 1)dx\Big]. \end{aligned} \tag{3.50}$$

From (3.23) and (3.26), we have

$$\partial(e_k(x)m_{21}(x, k)) = e_k(x)\bar{q}(x)m_{11}(x, k).$$

So, we get

$$\bar{\partial}(e_k(x)\overline{m_{21}(Q_1, x, -\bar{k})}) = e_k(x)q_1(x)\overline{m_{11}(Q_1, x, -\bar{k})} \tag{3.51}$$

and

$$\partial((e_k(x)m_{21}(Q_2, x, k)) = e_k(x)\bar{q}_2(x)m_{11}(Q_2, x, k) \tag{3.52}$$

Substituting (3.51) and (3.52) back into (3.50), then use integration by parts, and the fact $\sigma_i = 1$ on $\partial\Omega$, force the last four integrals of (3.50) to vanish. Thus, we get

$$I_1 = \frac{\pi}{k}(S_{\sigma_1})_{12}(-\bar{k}) + \frac{\pi}{\bar{k}}(S_{\sigma_2})_{21}(k). \tag{3.53}$$

We will now sketch the idea of why $I_2 = 0$. Let J_i be the i^{th} integral, $i = 1, 2, 3, 4$ of I_2, in the natural order. Notice

$$\int_{\mathbf{R}^2}\bar{\partial}(e_k(x)\overline{m_{21}(Q_1, x, -\bar{k})}\sigma_2^{1/2}(x))R(Q_2, x, k)dx = J_1 + J_3 \tag{3.54}$$

and

$$\int_{\mathbf{R}^2} \partial(e_k(x)m_{21}(Q_2,x,k)\sigma_1^{1/2})\overline{R(Q_1,x,-k)}dx = J_2 + J_4 \tag{3.55}$$

Consider the open ball $B_r \equiv B_r(0)$. Then integration by parts gives

$$\begin{aligned} I_2 = \lim_{r\to\infty}\Big(&\int_{\partial B_r} \frac{\bar{z}}{|z|} m_{21}(Q_2,x,k)e_k(x)\overline{R(Q_1,x,-k)}d\mu(x) \\ &- \int_{\partial B_r} \frac{\bar{z}}{|z|}\overline{m_{21}(Q_2,x,-k)}e_k(x)R(Q_2,x,k)d\mu(x)\Big) \end{aligned} \tag{3.56}$$

where r is sufficiently large so that for each $i = 1,2$ $supp(\sigma_i - 1) \subset B_r$.

Then we consider the asymptotic expansion of the Jost matrices in (3.55). From Proposition 2.18 of [Sun94a],

$$m_{21(Q,x,k)} = \frac{(S_\sigma)_{21}(k)}{|x|} + o(|x|^{-1}) \quad as\, |x| \to \infty$$

and a similar result holds for $m_{12}(x.k.)$

To complete Step 1, we will bound the first integral of (3.56) using the Trace Theorem and (3.40) as follows:

$$\begin{aligned} \left|\int_{\partial B_r} \frac{\bar{z}}{z} m_{21}(Q_2,x,k)e_k(x)\overline{R(Q_1,x,-k)}d\mu(x\right| &\le \frac{c(k)}{r}\int_{\partial B_r} |R(Q_1,x,-k))|\, d\mu(x) \\ &\le \frac{c(k)}{r^{1/q}}\, \|R(Q_1,x,-k)\|_{W^{1,q}(\mathbf{R}^2)}, \end{aligned} \tag{3.57}$$

where $q > 2$. The second integral of (3.56) can be treated similarly. Thus, we can see that $I_2 = 0$. This completes Step 1. Next, we will sketch some of the ideas for Step 2.

Step 2: Expand the derivatives in the integral I (3.45), use (3.41) and (3.42) and substitute $q_i = -\frac{\partial\sigma^{1/2}}{\sigma^{1/2}}$ back in the expression for I, we get after using integration by parts,

$$\begin{aligned} I = &-\int_{\partial\Omega} \sigma_1^{1/2}\sigma_2^{1/2}u(Q_2,x,k)\eta\partial\overline{u(Q_1,x,-k)}d\mu(x) \\ &+ \int_{\partial\Omega} \sigma_1^{1/2}\sigma_2^{1/2}\overline{u}(Q_1,x,-k)\bar{\eta}\bar{\partial}u(Q_2,x,k)d\mu(x) \end{aligned} \tag{3.58}$$

where we identify $\eta = \eta_1 + i\eta_2$ as (η_1, η_2) is the unit normal vector to $\partial\Omega$. We will let τ be the unit tangent vector to $\partial\Omega$, orthogonal to η. So $\tau = (\eta_2, -\eta_1)$ (we are assuming counterclockwise motion on the contour)

Let ∂_η be the normal derivative and ∂_τ be the tangential derivative. Then

$$\bar{\eta}\bar{\partial} = \frac{1}{2}\partial_\eta + \frac{i}{2}\partial_\tau \quad and \quad \eta\partial = \frac{1}{2}\partial_\eta - \frac{i}{2}\partial_\tau. \tag{3.59}$$

By (3.59) and $\Lambda_{\sigma_i} u = \sigma_i \partial_\eta u_i$, we get

$$\begin{aligned} I = & \frac{1}{2}\int_{\partial\Omega} \frac{\sigma_1^{1/2}}{\sigma_2^{1/2}} \overline{u(Q_1, x, -k)} \Lambda_{\sigma_2} u(Q_2, x, k) d\mu(x) \\ & - \frac{1}{2}\int_{\partial\Omega} \frac{\sigma_2^{1/2}}{\sigma_1^{1/2}} u(Q_2, x, k) \Lambda_{\sigma_1} \partial \overline{u(Q_1, x, -k)} d\mu(x) \\ & + \frac{i}{2}\int_{\partial\Omega} \sigma_1^{1/2}\sigma_2^{1/2} \partial_\tau (\overline{u(Q_1, x, -k)}) u(Q_2, x, k) d\mu(x). \end{aligned} \tag{3.60}$$

One can show the third term in the last expression vanishes. Using the fact that Λ_{σ_i} is self-adjoint, $\sigma_i = 1$ on $\partial\Omega$ and $\frac{\partial\sigma_i}{\partial\eta} = 0$, we get

$$I = \frac{1}{2}\int_{\partial\Omega} \overline{u(Q_1, x, -k)} (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) u(Q_2, x, k) d\mu(x) \tag{3.61}$$

We see that (3.44) follows from (3.46) □

The single layer potential S_k is a boundary integral operator defined by

$$S_k f(z) = \int_{\partial\Omega} f(y) G_k(z - y) d\mu(y). \tag{3.62}$$

where G_k is the Faddeev Green's function (3.6).

Theorem 8. *(Theorem 3.5.3 of [Knu02]) Let $\sigma \in W^{1,p}(\Omega)$ for $p > 2$ and suppose $\sigma = 1$ near $\partial\Omega$. Then for any $k \in \mathbf{C} - \{0\}$ the trace of the exponentially growing solution $u(\cdot, k)$ on $\partial\Omega$ is the unique solution to*

$$u(z, k) = \frac{1}{ik} e^{ikz} - S_k(\Lambda_\sigma - \Lambda_1) u(\cdot, k), \tag{3.63}$$

$k \in \mathbf{C} - \{0\}$, Λ_σ is the voltage-to-current map when Ω contains the conductivity distribution σ and Λ_1 is the Dirichlet-to-Neumann map of the homogeneous conductivity 1.

The boundary integral equation (3.63) is the first step of the reconstruction algorithm.

Equation (3.36) is not a pseudoanalytic equation (see Appendix A), so we will need the following result to show $M(z,k)$ is uniquely determined.

Lemma 9. *(Lemma 3.3.5 of [Knu02]) Define*

$$M^{\pm}(z,k) = m_{11}(z,k) \pm \overline{m_{12}(z,\bar{k})}, \tag{3.64}$$

where $z \in \boldsymbol{C}$ is fixed. Then the following d-bar equation holds

$$\bar{\partial}_k M^{\pm}(z,k) = \pm\overline{M^{\pm}(z,k)}e_{-k}(z)S_{21}(k), \tag{3.65}$$

where $S_{21}(k)$ is an off diagonal entry of S_σ.

Proof. This follows from (3.30), (3.31), (3.36), (3.22) and (3.23). □

Equation (3.65) is a pseudoanalytic equation, and we can apply Liouville's Theorem for pseudoanalytic function, this theorem can be found in the appendix (see Theorem 3.1.3 of [Knu02]) and conclude that (3.65) has at most one solution if $M^{\pm}(z,k) - 1 \in L^r(\mathbf{R}^2)$, $r > 2$. So it follows that $M(z,k)$ in (3.36) is uniquely determined.

We can recover Q from $S_\sigma(k)$ by using the following formula [BU97]

$$Q_\sigma(z) = \frac{1}{\pi} \lim_{k_0 \to \infty} \int_E D_k M(z,k) d\mu(k), \tag{3.66}$$

where $E = \{k : |k - k_0| < p\}$.

Solving the inverse conductivity problem for $\sigma \in W^{1,p}(\Omega)$ leads to a reconstruction algorithm. The following theorem is an important result of [Knu02].

Theorem 10. *(Theorem 3.4.1 of [Knu02]) Let Ω be open, bounded and smooth, and let $\sigma \in C^{1+\epsilon}(\bar{\Omega})$ for some $\epsilon > 0$. Then σ can be reconstructed uniquely from Λ_σ.*

There is slightly more regularity going on when $\sigma \in C^{1+\epsilon}(\bar{\Omega})$ compared to $\sigma \in W^{1,p}(\Omega)$. This extra smoothness is needed only when solving the pseudoanalytic equation (3.65). See Appendix A for a discussion on $C^{1+\epsilon}$.

Let $\widetilde{M}(z,k)$ be the unique solution to $D_k\widetilde{M}(z,k) = -Q^T\widetilde{M}(z,k)$ and define $\widetilde{M}^+(z,k)$ as in (3.64). Then it turns out that $\widetilde{M}^+(z,k)$ satifies the following d-bar equation (3.38) of [Knu02]

$$\bar{\partial}_k\widetilde{M}^+(z,k) = \overline{\widetilde{M}^+(z,k)}e_{-k}(z)\overline{(S_\sigma)_{21}(-k)}, \tag{3.67}$$

where $\widetilde{M}^+(z,k)$ is asymptotic to 1 for large $|k|$.

We can reconstruct $\widetilde{M}^+(z,k)$ from $S_\sigma(k)$. The conductivity σ then can be recovered from the equation (3.37) of [Knu02]

$$\sigma^{1/2}(z) = Re(\widetilde{M}^+(z,0)). \tag{3.68}$$

In summary, the main steps of the reconstruction algorithm for conductivities for $\sigma \in W^{1+\epsilon,p}(\bar{\Omega})$ are

(1) Compute u on $\partial\Omega$ using (3.63)

(2) Compute $(S_\sigma)_{21}(k)$ as in (3.44);

(3) Solve (3.67) for $\widetilde{M}^+(z,k)$;

(4) Reconstruct σ from (3.68).

3.3 Uniqueness For $\gamma \in W^{1,p}(\Omega)$

Francini came up with the first major result on the uniqueness of the inverse admittivity problem [Fra00]. Numerical results existed prior to this. She proved an analogous result of the uniqueness part of the inverse conductivity problem that Brown and Uhlmann established when conductivity $\sigma \in W^{1,p}(\Omega)$. We will now briefly discuss the result.

Let Ω be an open bounded domain in $\mathbf{R}^2$ with Lipschitz boundary. Let $\gamma = \sigma + i\omega\epsilon$, where σ and ϵ are measurable real-valued functions in Ω with σ bounded away from zero and infinity, ϵ is a small positive number.

We assume that there are two positive constants σ_0 and E such that

$$\sigma > \sigma_0 \;\; in \;\; \Omega, \tag{3.69}$$

$$\|\sigma\|_{W^{2,\infty}(\Omega)} \leq E, \quad \|\epsilon\|_{W^{2,\infty}(\Omega)} \leq E \tag{3.70}$$

and we can extend σ and ϵ to all of $\mathbf{R}^2$ so that $\sigma \equiv 1$ and $\epsilon \equiv 0$ outside a fixed ball containing Ω [Fra00].

The main result of [Fra00] is the following theorem

Theorem 11. *Let Ω be an open bounded domain in $\mathbf{R}^2$ with Lipschitz boundary. Let σ_j and ϵ_j satisfy the conditions given in* (3.69) *and* (3.70). *Then there exists a constant ω_0 such that if $\gamma_j = \sigma_j + i\omega\epsilon_j$ for j=1,2 and $\omega < \omega_0$ and if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

In this theorem, the imaginary part of γ is assumed to be small.

Francini's argument follows closely Brown and Uhlmann's proof when the conductivity $\sigma \in W^{1,p}(\Omega)$. She modifies parts of the proofs which break down when γ does not have real values and show that the method still works provided the imaginary part of γ is small.

Define the matrix potential Q_γ and matrix operator D by

$$Q_\gamma = \begin{pmatrix} 0 & -\frac{1}{2}\partial \log \gamma \\ -\frac{1}{2}\bar{\partial} \log \gamma & 0 \end{pmatrix}, \; D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}. \tag{3.71}$$

Equivalently we can write $-\frac{1}{2}\partial \log \gamma = -\frac{\partial \gamma^{1/2}}{\gamma^{1/2}}$ and $-\frac{1}{2}\bar{\partial} \log \gamma = -\frac{\bar{\partial}\gamma^{1/2}}{\gamma^{1/2}}$.

Throughout this section we will set

$$q = -\frac{1}{2}\partial \log \gamma$$

and

$$\widetilde{q} = -\frac{1}{2}\bar{\partial} \log \gamma.$$

So that

$$Q_\gamma = \begin{pmatrix} 0 & q \\ \widetilde{q} & 0 \end{pmatrix}.$$

She looks for exponential growing solutions of the elliptic system

$$\left(D - Q_\gamma\right) \psi = 0 \tag{3.72}$$

of the form

$$\psi(z,k) = M(z,k) \begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}, \tag{3.73}$$

where M is a 2×2 matrix valued function which converges to the identity matrix I as $|z| \to \infty$.

Let us specify the entries of the following matrix functions $\psi(z,k)$ and $M(z,k)$ by

$$\psi(z,k) = \begin{pmatrix} \psi_{11}(z,k) & \psi_{12}(z,k) \\ \psi_{21}(z,k) & \psi_{22}(z,k) \end{pmatrix}, \tag{3.74}$$

and

$$M(z,k) = \begin{pmatrix} m_{11}(z,k) & m_{12}(z,k) \\ m_{21}(z,k) & m_{22}(z,k), \end{pmatrix} \tag{3.75}$$

respectively. Then we have for $i = 1, 2$

$$m_{i1}(z,k) = \psi_{i1}(z,k)e^{-ikz} \quad \textit{and} \quad m_{i2}(z,k) = \psi_{i2}(z,k)e^{ik\bar{z}}. \tag{3.76}$$

An easy calculation shows that ψ satisfies equation (3.72) if and only if M satisfies

$$(D_k - Q_\gamma)M = 0 \text{ in } \mathbf{R}^2 \tag{3.77}$$

where D_k is the matrix operator defined by

$$D_k A = \begin{pmatrix} \bar{\partial} a_{11} & (\bar{\partial} - ik)a_{12} \\ (\partial + ik)a_{21} & \partial a_{22} \end{pmatrix} \tag{3.78}$$

and Q_γ is the matrix potential. The entries of (3.77) are given by

$$\bar{\partial} m_{11}(z,k) = q(z)m_{21}(z,k) \tag{3.79}$$

$$(\partial + ik)m_{21}(z,k) = \widetilde{q}(z)m_{11}(z,k) \tag{3.80}$$

$$(\bar{\partial} - ik)m_{12}(z,k) = q(z)m_{22}(z,k) \tag{3.81}$$

$$\partial m_{22}(z,k) = \widetilde{q}(z)m_{12}(z,k). \tag{3.82}$$

Again, we get the following simple but useful identities:

$$(\partial + ik)u = e_{-k}\partial(e_k u) \tag{3.83}$$

$$(\bar{\partial} - ik)u = e_{\bar{k}}\bar{\partial}(e_{-\bar{k}}u). \tag{3.84}$$

where $e_k \equiv e_k(\cdot)$.

A key result in [Fra00] is the following theorem.

Theorem 12. *(Theorem 3.1 of [Fra00]) Let σ and ϵ satisfy the conditions given in* (3.69) *and* (3.70)*, respectively. Then there exists a constant ω_0 such that for every $\omega < \omega_0$ and $k \in \boldsymbol{C}$ there is a unique solution $M(z,k)$ to* (3.77) *satisfying the condition*

$$M(\cdot, k) - I \in L^p(\boldsymbol{R}^2) \quad \textit{for some } p > 2.$$

This theorem is analogous to Theorem A parts (i) and (ii) of [BU97], but Francini has to approach the proof of this theorem differently since Q_γ is not Hermitian. The map $Q_\gamma \mapsto M - I$ is continuous with respect to the norm topologies on $L^p(\mathbf{R}^2)$.

The matrix potential Q_γ can be written as $Q_\gamma = Q_\sigma + i\omega Q'$, where

$$Q_\sigma = \begin{pmatrix} 0 & -\frac{1}{2}\partial \log \sigma \\ -\frac{1}{2}\bar{\partial} \log \sigma & 0 \end{pmatrix}$$

and

$$Q' = \frac{1}{2\sigma\gamma} \begin{pmatrix} 0 & \epsilon\partial\sigma - \sigma\partial\epsilon \\ \epsilon\bar{\partial}\sigma - \sigma\bar{\partial}\epsilon & 0 \end{pmatrix}.$$

To solve the problem

$$\begin{aligned} (D_k - Q_\gamma)M &= 0 \text{ in } \mathbf{R}^2, \\ M(\cdot, k) - I &\in L^p(\mathbf{R}^2), \ \textit{for some } p > 2, \end{aligned} \tag{3.85}$$

we look for solutions of the equation

$$(I - D_k^{-1} Q_\gamma)M = I.$$

Francini shows that the operator $I - D_k^{-1}Q_\gamma$ is invertible in a weighted Sobolev space (see Lemma 3.2 of [Fra00]). Francini defines the scattering transform $S_\gamma(k)$ of the matrix potential Q_γ by

$$S_\gamma(k) = \frac{i}{\pi} \int_{\mathbf{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} (Q_\gamma M)^{off}(z, k) d\mu(z), \tag{3.86}$$

A short calculation shows the scattering transform $S_\gamma(k)$ can be written

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R^2}} \begin{pmatrix} 0 & e^{-i\bar{k}z}q(z)\psi_{22}(z,k) \\ -e^{ik\bar{z}}\widetilde{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{3.87}$$

The following equivalent form of $S_\gamma(k)$ will also be use later on

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R^2}} \begin{pmatrix} 0 & q(z)e_{-\bar{k}}(z)m_{22}(z,k) \\ -\widetilde{q}(z)e_k(z)m_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{3.88}$$

A major result of [Fra00] is the following theorem.

Theorem 13. *(Theorem 4.1 of [Fra00]) Let σ and ϵ be as (3.69) and (3.70), respectively. Let M be as Theorem 12. Then the map $k \mapsto M(\cdot, k)$ is differentiable as a map into $L^r_{-\beta}(\boldsymbol{R}^2, M_2)$, and satisfies the d-bar equation*

$$\bar{\partial}_k M(z,k) = M(z,\bar{k})E_k(z)S_\gamma(k) \tag{3.89}$$

where $\bar{\partial}_k$ is the standard complex differential operator with respect to $\bar{k}$ and

$$E_k(z) = \begin{pmatrix} e_{\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix}.$$

Moreover for every $p > 2$, there exists a $K > 0$ such that

$$sup\, \|M(z,\cdot) - I\|_{L^p(\mathbf{R^2})} \leq K,$$

where K depends on the constants E, σ_0 and p.

The proof of this theorem is identical to Theorem A part (iv) of [BU97] since it does not use the assumption that σ has to be real. The scattering transform S_γ is related to the scattering transform S_σ in the following simple way:

$$S_\gamma(k) = S_\sigma(k) + \omega S'(k), \tag{3.90}$$

where $S'(k) = \frac{1}{\omega}(S_\gamma(k) - S_\sigma(k))$. The form of $S'(k)$ is

$$S'(k) = \frac{i}{\pi} \int_{\mathbf{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & e_{-k}(z) \end{pmatrix} (Q_\sigma M_1 + iQ_\gamma M)^{off}(z,k) d\mu(z).$$

Francini proved the following theorem [Fra00] for the recovery of Q_γ.

Theorem 14. *(Theorem 6.2 of [Fra00]) For any $p > 0$,*

$$Q_\gamma(z) = \lim_{k_0 \to \infty} \mu(B_p(0))^{-1} \int_E D_k M(z,k) d\mu(k),$$

where $E = \{k : |k - k_0| < p\}$.

In chapter six, we will establish some properties involving exponentially growing solutions that we believe will be use for a reconstruction algorithm for $\gamma \in W^{1,p}(\Omega)$ following the ideas of several papers for the the reconstruction of $\sigma \in W^{1,p}(\Omega)$, from the Dirichlet-to-Neumann map.

3.4 Another Approach to $\gamma \in W^{1,p}(\Omega)$

In the previous section, we sketched the ideas Francini used for an analogous result of the uniqueness part of the inverse conductivity problem that Brown and Uhlmann established when conductivity $\sigma \in W^{1,p}(\Omega)$. In [Sun94a] and [BBR01], the scattering method was used to analyze the first order elliptic system hold with hermitic condition on the off-diagonal entries for the matrix potential. More precisely, the idea of this approach is to solve a more general first order elliptic system with the matrix potential Q_γ replaced by a Hermitian matrix R, given by

$$R = \begin{pmatrix} 0 & \rho \\ \bar{\rho} & 0 \end{pmatrix}. \tag{3.91}$$

The potential matrix Q_γ in (3.71) is not Hermitian and we end up not having any symmetries in the entries of $M(z,k)$ and the off-diagonal entries of the scattering transform $S_\gamma(k)$. In this section, we will define a Q_γ that preserves the uniqueness of the inverse admittivity problem and also gives us some symmetry relations as we will see in chapter five.

Let Ω be an open bounded domain in $\mathbf{R}^2$ with Lipschitz boundary. Let $\gamma = \sigma + i\omega\epsilon$, where σ and ϵ are measurable real-valued functions in Ω with σ bounded away from zero and infinity, ϵ is a small positive number.

We assume that there are positive constants such similar bounds as (3.69) and (3.70) hold.

We can extend σ and ϵ to all of $\mathbf{R}^2$ so that $\sigma \equiv 1$ and $\epsilon \equiv 0$ outside a fixed ball containing Ω using an argument similar to [Fra00].

All the major results of [Fra00] will hold provided the imaginary part is assumed to be small.

Theorem 15. *Let Ω be an open bounded domain in $\mathbf{R}^2$ with Lipschitz boundary. Let σ_j and ϵ_j satisfy the conditions given in* (3.69) *and* (3.70). *Then there exists a constant ω_0 such that if $\gamma_j = \sigma_j + i\omega\epsilon_j$ for j=1,2 and $\omega < \omega_0$ and if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

Define the matrix potential Q_γ and matrix operator D by

$$Q_\gamma = \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}, \; D = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix}. \tag{3.92}$$

where $q = -\frac{1}{2}\partial \log \gamma$ and $\bar{q} = -\frac{1}{2}\bar{\partial} \log \bar{\gamma}$.

We can see that Q_γ is Hermitian. Equivalently we can write $q = -\frac{\partial \gamma^{1/2}}{\gamma^{1/2}}$ and $\bar{q} = -\frac{\bar{\partial}\bar{\gamma}^{1/2}}{\bar{\gamma}^{1/2}}$.

Throughout this section we will consistently set

$$q = -\frac{1}{2}\partial \log \gamma$$

and

$$\bar{q} = -\frac{1}{2}\bar{\partial}\log\bar{\gamma}.$$

We looks for exponential growing solutions of the elliptic system

$$\left(D - Q_\gamma\right)\psi = 0 \tag{3.93}$$

of the form

$$\psi(z,k) = M(z,k)\begin{pmatrix} e^{izk} & 0 \\ 0 & e^{-i\bar{z}k} \end{pmatrix}, \tag{3.94}$$

where M is a 2×2 matrix valued function which converges to the identity matrix I as $|z| \to \infty$.

Let us set the following matrix functions $\psi(z,k)$ and $M(z,k)$ by

$$\psi(z,k) = \begin{pmatrix} \psi_{11}(z,k) & \psi_{12}(z,k) \\ \psi_{21}(z,k) & \psi_{22}(z,k) \end{pmatrix}, \tag{3.95}$$

and

$$M(z,k) = \begin{pmatrix} m_{11}(z,k) & m_{12}(z,k) \\ m_{21}(z,k) & m_{22}(z,k), \end{pmatrix} \tag{3.96}$$

respectively.

So we have for $i = 1, 2$

$$m_{i1}(z,k) = \psi_{i1}(z,k)e^{-ikz} \quad and \quad m_{i2}(z,k) = \psi_{i2}(z,k)e^{ik\bar{z}}. \tag{3.97}$$

A trivial calculation shows that ψ satisfies equation (3.93) if and only if M satisfies

$$(D_k - Q_\gamma)M = 0, \ in \ \mathbf{R}^2 \tag{3.98}$$

where D_k is the matrix operator defined by

$$D_k A = \begin{pmatrix} \bar{\partial}a_{11} & (\bar{\partial} - ik)a_{12} \\ (\partial + ik)a_{21} & \partial a_{22} \end{pmatrix} \tag{3.99}$$

and Q_γ is the matrix potential.

The entries of 3.98 are given by

$$\begin{aligned}\bar{\partial} m_{11}(z,k) &= q(z) m_{21}(z,k) &(3.100)\\ (\partial + ik) m_{21}(z,k) &= \bar{q}(z) m_{11}(z,k) &(3.101)\\ (\bar{\partial} - ik) m_{12}(z,k) &= q(z) m_{22}(z,k) &(3.102)\\ \partial m_{22}(z,k) &= \bar{q}(z) m_{12}(z,k). &(3.103)\end{aligned}$$

Again, we get the following simple but useful identities:

$$\begin{aligned}(\partial + ik)u &= e_{-k}\partial(e_k u) &(3.104)\\ (\bar{\partial} - ik)u &= e_{\bar{k}}\bar{\partial}(e_{-\bar{k}} u). &(3.105)\end{aligned}$$

where $e_k \equiv e_k(\cdot)$.

Define the scattering transform $S_\gamma(k)$ of the matrix potential Q_γ by

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} e_{-\bar{k}}(z) & 0 \\ 0 & -e_k(z) \end{pmatrix} (Q_\gamma M)^{off}(z,k) d\mu(z). \tag{3.106}$$

A short calculation shows the scattering transform $S_\gamma(k)$ can be written

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} 0 & e^{-i\bar{k}z} q(z)\psi_{22}(z,k) \\ -e^{ik\bar{z}}\bar{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{3.107}$$

The following equivalent form of $S_\gamma(k)$ will also be use later on

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} 0 & q(z) e_{-\bar{k}} m_{22}(z,k) \\ -\bar{q}(z) e_k(z) m_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{3.108}$$

Again, we get a similar d-bar equation as in (3.89).

Chapter 4

THE SCATTERING TRANSFORM USED IN THE INVERSE CONDUCTIVITY PROBLEM

Earlier we saw that both the inverse and the direct scattering transforms are key ingredients in solving the inverse admittivity (conductivity) problem for conductivities in the Sobolev spaces $W^{k,p}(\Omega)$ with $k = 1, 2$ and for admittivities in $W^{1,p}(\Omega)$.

It is important to understand the behaviors of the non-physical scattering transform $\mathbf{t}(k)$ for both large and small values of $|k|$ since they play a role in the numerical solutions of the d-bar equation [SMI00]. Other properties of the scattering transform such as symmetry properties also are important. In this section we will discuss the known properties of the scattering transforms $\mathbf{t}(k)$ and $S_\sigma(k)$.

Nachman shows that $\mathbf{t}(k)$ is approximately the Fourier transform of q for large $|k|$. However, there are some qualitative similaries between $\mathbf{t}(k)$ and the Fourier transform of the potential q. Nachman also came up with some estimates for $\mathbf{t}(k)$ when $|k|$ is small and large. Siltanen [Sil99] improved the bounds for $\mathbf{t}(k)$ when $|k|$ is small and large. Siltanen came up with several symmetry properties between conductivity σ and $\mathbf{t}(k)$. Siltanen shows (roughly) that

(1) $\mathbf{t}(k)$ is radial if and only if σ is radial

(2) Dilating the conductivity $\sigma(\lambda x)$ for any $\lambda > 0$ corresponds to dilating the scattering transform by $\mathbf{t}(k/\lambda)$;

(3) Reflectional symmetry in the conductivity σ corresponds to reflectional symmetry in $\mathbf{t}(k)$;

(4) Translating the conductivity σ corresponds to multiplication of $\mathbf{t}(k)$ by a certain exponential function of modulus one.

There are few known properties for the scattering transform $S_\sigma(k)$. However Knudsen [Knu02] and J. Barceló, T. Barceló and Ruiz [BBR01] proved some properties of the scattering transform $S_\sigma(k)$.

In section 4.1, we will discuss some of the properties of the non-physical scattering transform $\mathbf{t}(k)$ that Nachman [Nac96] and Siltanen [Sil99] have derived. In section 4.2 we will briefly discuss the known properties of $S_\sigma(k)$ that Knudsen [Knu02] and J. Barceló, T. Barceló and Ruiz [BBR01].

4.1 Properties of the Scattering Transform $\mathbf{t}(k)$

In this section we will discuss some properties of the scattering transform $\mathbf{t}(k)$.

In two dimensions, one can show that if (roughly) the potential q comes from a conductivity σ, then there are no exceptional points [Nac96]. That is, there exist exponentially growing solutions $\psi(z,k)$, for all $k \in \mathbf{C} - (0)$ which satisfy (3.1). So we can define the non-physical scattering transform $\mathbf{t}(k) = \int_\Omega e_k(z) q(z) \psi(z,k) dz$, $k \in \mathbf{C} - \{0\}$.

Nachman shows that for large $|k|$ $\mathbf{t}(k)$ is approximately the Fourier transform of the potential $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. There are some qualitative similarities between

$\mathbf{t}(k)$ and the Fourier transform of the potential q. Nachman also proved some estimates for $\mathbf{t}(k)$ when $|k|$ is small and large. Let us briefly discuss these properties.

Nachman [Nac96] proved that for some $s \in (-1, 0)$ and large $|k|$,

$$|\mathbf{t}(k_1, k_2) - \hat{q}(-2k_1, 2k_2)| \leq C|k|^s$$

for some constant C. That is, $\mathbf{t}(k)$ is approximately the Fourier transform of q evaluated at the point $(-2k_1, 2k_2) \in \mathbf{R}^2$ for large $|k|$. Nachman also showed that $\mathbf{t}(k)$ is continuous outside the origin.

Siltanen proved an estimate of $\mathbf{t}(k)$ for large $|k|$ which is a consequence of the smoothness of σ.

Theorem 16. *(Theorem 3.13 of [Sil99]) Let $\Omega \subset \mathbf{R}^2$ be a bounded simply connected C^∞ domain. Let $\sigma \in C^{2+m}(\Omega)$, $m \geq 1$, have a positive lower bound and assume $\sigma \equiv 1$ near $\partial\Omega$. Define $q = \sigma^{-1/2}\Delta\sigma^{1/2}$ and $\mathbf{t}(k)$ be given as in* (3.4)*. Then there exists a constant C such that $|\mathbf{t}(k)| \leq C|k|^{-m}$ for large $|k|$.*

Siltanen also sharpened the estimate of $\mathbf{t}(k)$ when k is small.

Theorem 17. *(Theorem 3.18 of [Sil99]) Let $\Omega \subset \mathbf{R}^2$ be a bounded simply connected C^∞ domain. Let $\sigma \in W^{2,p}(\Omega)$, $1 < p < 2$, $m \geq 1$, have a positive lower bound and assume $\sigma \equiv 1$ near $\partial\Omega$. Let $\mathbf{t}(k)$ be given as in* (3.4)*. Then there exists a constant C such that $|\mathbf{t}(k)| \leq C|k|^{1+\epsilon}$ for small $|k|$ and for all $0 < \epsilon < 2/p'$.*

We will now discuss some of the symmetry properties between the scattering transform $\mathbf{t}(k)$ and the conductivity $\sigma \in W^{2,p}(\Omega)$.

Theorem 18. *(Theorem 3.19 of [Sil99]) Let $\Omega \subset \mathbf{R}^2$ be a bounded simply connected C^∞ domain. Let $\sigma_i \in C^2$ have a positive lower bound and satisfy $\sigma_i - 1 \in C_0^\infty(\Omega)$*

for i=1,2. Set $q_i = {\sigma_i}^{-1/2}\Delta\sigma_i^{1/2}$ *and denote by* $\mathbf{t}_i$ *the non-physical scattering transforms correponding to* q_i.

Let $\vartheta \in \mathbf{R}$, $\lambda > 0$ *and* $x' \in \mathbf{R}^2$; *denote* $z_\vartheta := e^{i\vartheta}(z_1 + iz_2)$ *for any* $z \in \mathbf{R}^2$. *Then the following four equivalences hold:*

(1) $\sigma_2(x) = \sigma_1(x_{-\vartheta})\ \forall\ x \in \mathbf{R}^2 \Leftrightarrow \mathbf{t_2}(k) = \mathbf{t_1}(k_\vartheta)\ \forall\ k \in \mathbf{C}$;

(2) $\sigma_2(x) = \sigma_1(\lambda x)\ \forall\ x \in \mathbf{R}^2 \Leftrightarrow \mathbf{t_2}(k) = \mathbf{t_1}(\lambda^{-1}k)\ \forall\ k \in \mathbf{C}$;

(3) $\sigma_2(x) = \sigma_1(x_{-\bar{x}})\ \forall\ x \in \mathbf{R}^2 \Leftrightarrow \mathbf{t_2}(k) = \overline{\mathbf{t_1}(\bar{k})}\ \forall\ k \in \mathbf{C}$;

(4) $\sigma_2(x) = \sigma_1(x - x')\ \forall\ x \in \mathbf{R}^2 \Leftrightarrow \mathbf{t_2}(k) = e_k(x')\mathbf{t_1}(k)\ \forall\ k \in \mathbf{C}$.

Mueller and Siltanen [MS03] establish the following result.

Theorem 19. *Assume that* $\mathbf{t}_j(k)/\bar{k} \in L^p(\mathbf{R}^2)$ *for all* $p \in (2-\epsilon, 2+\epsilon)$ *for some* $\epsilon > 0$ *and j=1,2. Let* $\sigma_j \in C(\mathbf{R}^2)$ *be given by the* $\bar{\partial}$ *inversion from the function* $\mathbf{t}_j$ *as described in theorem 4.1 of [Nac96]. Let* $\vartheta \in \mathbf{R}$, $\lambda > 0$ *and* $x' \in \mathbf{R}^2$; *denote* $z_\vartheta := e^{i\vartheta}(z_1 + iz_2)$ *for any* $z \in \mathbf{R}^2$.

(1) $\mathbf{t_2}(k) = \mathbf{t_1}(k_{-\vartheta})\ \forall\ k \in \mathbf{C} \Rightarrow \sigma_2(x) = \sigma_1(x_\vartheta)\ \forall\ x \in \mathbf{R}^2$;

(2) $\mathbf{t_2}(k) = \mathbf{t_1}(\lambda^{-1}k)\ \forall\ k \in \mathbf{C} \Rightarrow \sigma_2(x) = \sigma_1(\lambda x)\ \forall\ x \in \mathbf{R}^2$;

(3) $\mathbf{t_2}(k) = \overline{\mathbf{t_1}(\bar{k})}\ \forall\ k \in \mathbf{C}; \Rightarrow \sigma_2(x) = \sigma_1(x_{-\bar{x}})\ \forall\ x \in \mathbf{R}^2$;

(4) $\mathbf{t_2}(k) = e_k(x')\mathbf{t_1}(k)\ \forall\ k \in \mathbf{C} \Rightarrow \sigma_2(x) = \sigma_1(x - x')\ \forall\ x \in \mathbf{R}^2$.

4.2 Properties of the Scattering Transform for the First Order Elliptic System

There is a well developed theory behind the scattering transform $\mathbf{t}(k)$, but there is not much known about the scattering transform $S_\sigma(k)$ in the real case. We will now briefly discuss the known properties of $S_\sigma(k)$ and some of its off-diagmonal entries .

Let's recall that we consider the scattering transform $S_\sigma(k)$ for the first order elliptic system in the plane given by

$$(\begin{pmatrix} \bar{\partial} & 0 \\ 0 & \partial \end{pmatrix} - \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix})\psi = 0, \tag{4.1}$$

where $S_\sigma(k)$ is given as

$$S_\sigma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} 0 & e^{-i\bar{k}z}q(z)\psi_{22}(z,k) \\ -e^{ik\bar{z}}\bar{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z) \tag{4.2}$$

One beautiful result of the scattering theory as developed by Beals and Coifman [BC88] is a version of the nonlinear Plancherel identity relating the scattering data $S_\sigma(k)$ and the matrix potential Q_σ,

$$\int |Q_\sigma|^2 d\mu = \int |S_\sigma|^2 d\mu, \tag{4.3}$$

where $Q_\sigma = Q_\sigma^*$.

This identity (4.3) does not imply the continuity of the map $Q_\sigma \to S_\sigma$ since the map is nonlinear. This identity was used in the proof of the uniqueness problem when conductivity $\sigma \in W^{1,p}(\Omega)$.

J. Barceló, T. Barceló and Ruiz [BBR01] have established continuity of the map $Q_\sigma \to S_\sigma$ when the matrix potential Q_σ is Hölder continuous and compactly supported. This was a positive step forward in establishing the continuous dependence of the conductivity $\sigma \in W^{1,p}(\Omega)$ on the Dirichlet-to-Neumann map Λ_σ.

Brown [Bro01] proved some L^2 estimates for the scattering transform $S_\sigma(k)$ in two dimensions. The key result from this paper is that the scattering map $Q_\sigma \to S_\sigma$ is Lipschitz continuous on a neighborhood of zero in L^2, where Q_σ is small.

Let $(S_\sigma)_{12}(k)$ and $(S_\sigma)_{21}(k)$ be the off diagonal entries of the scattering transform $S_\sigma(k)$. In [BBR01], the authors mention that since we have (3.31) and Q_σ is Hermitian, $\overline{(S_\sigma)_{12}(k)} = (S_\sigma)_{21}(\bar{k})$.

Knudsen and Tamasan [KT04] proved that if $\widetilde{S_\sigma}(k)$ is the scattering transform of $\tilde{q} = -\bar{q}$, then $\widetilde{S_\sigma}(k) = \overline{S_\sigma(-k)}$.

Knudsen [Knu02] proves the following result which is analogous to the result that radial symmetry in the conductivity implies symmetry in the scattering transform $\mathbf{t}(k)$ (Theorem 18). Let $\sigma \in W^{1,p}(\Omega)$, $p > 2$ and assume $\sigma(z) = \sigma(e^{i\theta}z)$ for some angle θ. Then

$$e^{i\theta}(S_\sigma)_{21}(e^{i\theta}k) = (S_\sigma)_{21}(k) \text{ and } e^{-i\theta}(S_\sigma)_{12}(e^{i\theta}k) = (S_\sigma)_{12}(k). \tag{4.4}$$

The following properties of the scattering transform $S_\sigma(k)$ are stated as Proposition 4.2 [Fra00], and they were proved by Sung [Sun94b]:

If $\|\sigma\|_{W^{1,\infty}(\Omega)} \leq E$ for some positive constant E, then

$$S_\sigma(k) \in L^2(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$$

and if there exists a constant E such that $\|\sigma\|_{W^{2,\infty}(\Omega)} \leq E$, then

$$kS_\sigma \in L^2(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2).$$

Francini proves the following result (see Lemma 4.3 of [Fra00]) : Let S' be as in (3.90). Then there is a constant C such that

$$sup_{k \in \mathbf{R}^2} |kS'(k)| \leq C$$

for every $\omega < \omega_0$.

Chapter 5

SOME NEW PROPERTIES OF THE SCATTERING TRANSFORM $S_\gamma(K)$

In this chapter, we will discuss several new properties of the two-dimensional scattering transform $S_\gamma(k)$ used in the first order elliptic system for the admittivity case.

Throughout this chapter, we are using the potential matrix Q_γ as defined in (3.92) unless we stated otherwise. Recall that

$$q = -\frac{1}{2}\partial \log \gamma \quad and \quad \bar{q} = -\frac{1}{2}\bar{\partial} \log \bar{\gamma}.$$

Francini's definition of Q_γ as defined in (3.71) will not lead to any symmetries between the entries of the Jost matrix $M(z, k)$, and thus, there won't be symmetries between the off-diagonal entries of the scattering transform $S_\gamma(k)$ since Q_γ is not Hermitian.

As we have seen, when the conductivity $\sigma \in W^{2,p}(\Omega)$, the scattering transform $\mathbf{t}(k)$ plays a major role in solving the inverse conductivity problem. In this context, Nachman [Nac96] and Siltanen [Sil99] have developed some properties for the scattering transform $\mathbf{t}$. Moreover, applications of the properties of $\mathbf{t}$ include verification of numerical results of the reconstruction algorithm, reduction of computation time and derivation of approximations [SMI00] and [MS03].

We have also seen that the scattering transform $S_\sigma(k)$ used by Brown and Uhlmann [BU97] was used in the argument to solve the inverse problem for $\sigma \in W^{1,p}(\Omega)$. It relies on scattering theory that had been developed earlier by Beals and Coifman [BC88] and Sung [Sun94a], [Sun94b], [Sun94c]. Knudsen [Knu02], Brown [Bro01], J. Barceló, T. Barceló and Ruiz [BBR01] established some properties for the scattering transform $S_\sigma(k)$. Knudsen and Tamasan [KT04] and [Knu02], came up with a reconstruction algorithm with conductivity σ in the Sobolev space $W^{1+\epsilon,p}(\Omega)$ with $p > 2$ and $0 < \epsilon < 1$.

Francini [Fra00] used some properties for the scattering transform $S_\sigma(k)$, which played a role in solving the inverse admittivity problem with $\gamma \in W^{1,p}(\Omega)$. These properties of the scattering transform $S_\sigma(k)$ were proved by Sung [Sun94b].

The scattering transform $S_\gamma(k)$ which appears in the d-bar equation is an off-diagonal matrix valued function.

The properties of the two dimensional scattering transform $S_\gamma(k)$ that we establish are important from the point of view of numerical work that others in this area can use. More precisely, applications of new properties of the non-physical scattering transform $S_\gamma(k)$ include:

(1) Verification of numerical results of the reconstruction algorithm;

(2) Reduction of computation;

(3) Derivation of approximations;

(4) Use in the evolution equations.

In the first section we will show that the non-physical scattering transform $S_\gamma(k)$ is related to the Fourier transform of $q = -\frac{1}{2}\partial \log \gamma = -\frac{\partial \gamma^{1/2}}{\gamma^{1/2}}$. for large $|k|$. This will holds if we use either (3.71) or (3.92) for Q_γ.

We will prove that if Q_γ is defined as in (3.92), then we get the following symmetries among the entries of the Jost matrix $M(z,k)$,

$$m_{11}(z,k) = \overline{m_{22}(z,\bar{k})} \quad m_{12}(z,k) = \overline{m_{21}(z,\bar{k})}.$$

Next we will establish several symmetry properties of the off diagonal entries of $S_\gamma(k)$.

More precisely, we will show that if the Q_γ is as defined in (3.92), then

$$\overline{(S_\gamma)_{12}(k)} = (S_\gamma)_{21}(\bar{k}),$$

We will also show that the following identitities hold

$$(S_\gamma)_{21}(e^{i\theta}k) = e^{-i\theta}(S_\gamma)_{21}(k) \quad \text{and} \quad (S_\gamma)_{12}(e^{i\theta}k) = e^{i\theta}(S_\gamma)_{12}(k)$$

under certain conditions. These identities will hold if we use Q_γ as defined as in (3.92).

5.1 The Scattering Transform $S_\gamma(k)$ Behaves Like a Fourier Transform of q

Nachman shows that for large $|k|$ $\mathbf{t}(k)$ is approximately the Fourier transform of the potential $q = \sigma^{-1/2}\Delta\sigma^{1/2}$.

In this section, we will see that the scattering transform $S_\gamma(k)$ behaves like the Fourier transform of q for large $|k|$. This will holds regardless if Q_γ is as defined in (3.71) or (3.92). We will prove it for the case if Q_γ is as (3.92).

Let us recall that we define $q = -\frac{1}{2}\partial \log \gamma$. and $\bar{q} = -\frac{1}{2}\bar{\partial} \log \bar{\gamma}$.

Also recall that the scattering transform $S_\gamma(k)$ is defined by

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2} \begin{pmatrix} 0 & e^{-i\bar{k}z}q(z)\psi_{22}(z,k) \\ -e^{i\bar{z}\bar{k}}\bar{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z). \tag{5.1}$$

Also, let us recall that

$$\psi(z,k) \backsim \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ik\bar{z}} \end{pmatrix} \tag{5.2}$$

for large $|k|$.

It follows that for large $|k|$,

$$\begin{aligned} S_\gamma(k) &\backsim \frac{i}{\pi}\int_{\mathbf{C}} \begin{pmatrix} 0 & e^{-i(\bar{k}z+k\bar{z})}q(z) \\ -e^{i(kz+\bar{k}\bar{z})}\bar{q}(z) & 0 \end{pmatrix} d\mu(z), \\ &= \frac{i}{\pi}\int_{\mathbf{C}} \begin{pmatrix} 0 & e^{-i(\bar{k}z+k\bar{z})}q(z) \\ -e^{i(kz+\bar{k}\bar{z})}\bar{q}(z) & 0 \end{pmatrix} d\mu(z), \\ &= \frac{i}{\pi}\int_{\mathbf{C}} \begin{pmatrix} 0 & e^{-2iRe(k\bar{z})}q(z) \\ -e^{2iRe(kz)}\bar{q}(z) & 0 \end{pmatrix} d\mu(z), \\ &= \frac{i}{\pi}\int_{\mathbf{C}} \begin{pmatrix} 0 & e^{-i(2k_1,2k_2)\cdot(z_1,z_2)}q(z) \\ -e^{-i(-2k_1,2k_2)\cdot(z_1,z_2)}\bar{q}(z) & 0 \end{pmatrix} d\mu(z), \end{aligned} \tag{5.3}$$

Thus, for large $|k|$

$$S_\gamma(k) \backsim 2i \begin{pmatrix} 0 & \hat{q}(2k_1,2k_2) \\ -\hat{\bar{q}}(-2k_1,2k_2) & 0 \end{pmatrix} \tag{5.4}$$

That is, $S_\gamma(k)$ is related to the Fourier transform q for large $|k|$.

So we can expect some properties of the Fourier transform of q to carry over for $S_\gamma(k)$ for large $|k|$.

5.2 Some Symmetry Properties of Off-Diagonal Entries Of The Scattering Transform $S_\gamma(k)$

In Nachman's case, if we have symmetries in the scattering transform then we have symmetries in the conductivity [MS03].

In this section we will show that there are some symmetry properties between $(S_\gamma)_{12}(k)$ and $(S_\gamma)_{21}(k)$, the off-diagonal entries of the scattering transform $S_\gamma(k)$.

Let $(S_\sigma)_{12}(k)$ and $(S_\sigma)_{21}(k)$ be the off diagonal entries of the scattering transform $S_\sigma(k)$. In [BBR01], the authors mention that since we have the uniqueness of $M(z,k)$ and there are certain symmetry relations of the entries of $M(z,k)$, we get

$$\overline{(S_\sigma)_{12}(k)} = (S_\sigma)_{21}(\bar{k}).$$

We will first establish that there are indeed symmetry relations for the entries of $M(z,k)$. First, we will need to recall the following result. Let $\gamma(z) = \sigma(z) + i\omega\epsilon(z) \in W^{1,p}(\Omega)$ where the $W^{1,p}(\Omega)$ norms of σ and ϵ are bounded. Then there is a unique solution $M(z,k)$ to (3.98) such that $M(\cdot,k) - I \in L^p(\mathbf{R^2})$ for some $p > 2$.

Throughout this section, we will consistently use the following definitions of q and $\bar{q}$. Let $q = -\frac{1}{2}\partial \log \gamma$, and $\bar{q}$ will denotes the complex conjugate of q.

Before we establish a key lemma that allows us to discuss symmetry properties of the off-diagonal entries of the scattering transform $S_\gamma(k)$, we will need the following equalities which were mentioned in section 4.3, but are restated here for convenience. The entries of (3.98) are given by

$$\begin{aligned}
\bar{\partial} m_{11}(z,k) &= q(z) m_{21}(z,k) && (5.5)\\
(\partial + ik) m_{21}(z,k) &= \bar{q}(z) m_{11}(z,k) && (5.6)\\
(\bar{\partial} - ik) m_{12}(z,k) &= q(z) m_{22}(z,k) && (5.7)\\
\partial m_{22}(z,k) &= \bar{q}(z) m_{12}(z,k). && (5.8)
\end{aligned}$$

and the following simple, but useful identities

$$\begin{aligned}
(\partial + ik) u &= e_{-k} \partial (e_k u) && (5.9)\\
(\bar{\partial} - ik) u &= e_{\bar{k}} \bar{\partial} (e_{-\bar{k}} u). && (5.10)
\end{aligned}$$

where $e_k \equiv e_k(\cdot)$.

Let us recall the Cauchy transforms, denoted by ∂^{-1} and $\bar{\partial}^{-1}$

$$\bar{\partial}^{-1} f(z) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(w)}{z - w} d\mu(w) \tag{5.11}$$

and

$$\partial^{-1} f(z) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(w)}{\bar{z} - \bar{w}} d\mu(w), \tag{5.12}$$

where $d\mu$ is the Lebesgue measure in $\mathbf{R}^2$

We will now establish an important lemma that enables us to establish several symmetry properties of the off-diagonal entries $(S_\gamma)_{12}(k)$ and $(S_\gamma)_{21}(k)$ of the scattering transform $S_\gamma(k)$.

Lemma 20. *Let $\gamma(z) = \sigma(z) + i\epsilon(z) \in W^{1,p}(\Omega)$, where σ and ϵ satisfy* (3.69) *and* (3.70)*. Then we have the following identities,*

$$m_{11}(z,k) = \overline{m_{22}(z,\bar{k})} \quad \textit{and} \quad m_{21}(z,k) = \overline{m_{12}(z,\bar{k})}.$$

Proof. By (5.9),

$$(\partial + ik) m_{21}(x,k) = e_{-k}(x) \partial (e_k(x) m_{21}(x,k)). \tag{5.13}$$

So from (5.13) and (5.6) we get

$$\partial (e_k(x) m_{21}(x,k)) = e_k(x) \bar{q}(x) m_{11}(x,k). \tag{5.14}$$

We can solve for $m_{11}(x,k)$ in (5.5) using (5.11), we get

$$m_{11}(x,k) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{q(z) m_{21}(z,k)}{x - z} \, dz_1 \, dz_2. \tag{5.15}$$

We can solve for $m_{21}(x,k)$ in (5.14) using (5.12), we get

$$m_{21}(x,k) = \frac{e_{-k}(x)}{\pi} \int_{\mathbf{C}} \frac{e_k(z) \bar{q}(z) m_{11}(z,k)}{\bar{x} - \bar{z}} \, dz_1 \, dz_2 \tag{5.16}$$

Substituting (5.16) into (5.15) yields

$$m_{11}(x,k) = \frac{1}{\pi^2}\int_{\mathbf{C}} \frac{e_{-k}(z)q(z)}{x-z}\int_{\mathbf{C}} \frac{e_k(\xi)\bar{q}(\xi)m_{11}(\xi,k)}{\bar{z}-\bar{\xi}}\,d\xi_1\,d\xi_2\,dz_1\,dz_2 \tag{5.17}$$

Thus, from (5.17) we get

$$\overline{m_{11}(x,\bar{k})} = \frac{1}{\pi^2}\int_{\mathbf{C}} \frac{\overline{e_{-\bar{k}}(z)\,q(z)}}{\bar{x}-\bar{z}}\int_{\mathbf{C}} \frac{\overline{e_{\bar{k}}(\xi)}q(\xi)\overline{m_{11}(\xi,\bar{k})}}{z-\xi}\,d\xi_1\,d\xi_2\,dz_1\,dz_2 \tag{5.18}$$

From (5.10), we get

$$(\bar{\partial}-ik)m_{12}(x,k) = e_{\bar{k}}(x)\bar{\partial}(e_{-\bar{k}}(x)m_{12}(x,k)) = q(x)m_{22}(x,k) \tag{5.19}$$

Similarly, solving for $m_{22}(x,k)$ from (5.8) and $m_{12}(x,k)$ from (5.19), we get

$$m_{22}(x,k) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\bar{q}(z)m_{12}(z,k)}{\bar{x}-\bar{z}}\,dz_1\,dz_2 \tag{5.20}$$

and

$$m_{12}(x,k) = \frac{e_{\bar{k}}(x)}{\pi}\int_{\mathbf{C}} \frac{e_{-\bar{k}}(z)q(z)m_{22}(z,k)}{x-z}\,dz_1\,dz_2. \tag{5.21}$$

Substituting (5.21) into (5.20) yield

$$\begin{aligned} m_{22}(x,k) &= = \frac{1}{\pi^2}\int_{\mathbf{C}} \frac{e_{\bar{k}}(z)\bar{q}(z)}{\bar{x}-\bar{z}}\int_{\mathbf{C}} \frac{e_{-\bar{k}}(\xi)q(\xi)m_{22}(\xi,k)}{z-\xi}\,d\xi_1\,d\xi_2\,dz_1\,dz_2, \\ &= \frac{1}{\pi^2}\int_{\mathbf{C}} \frac{\overline{e_{-\bar{k}}(z)\,q(z)}}{\bar{x}-\bar{z}}\int_{\mathbf{C}} \frac{\overline{e_{\bar{k}}(\xi)}q(\xi)m_{22}(\xi,\bar{k})}{z-\xi}\,d\xi_1\,d\xi_2\,dz_1\,dz_2. \end{aligned} \tag{5.22}$$

Thus, by the uniqueness of $M(x,k)$, equations (5.18) and (5.22), we have $m_{11}(z,k) = \overline{m_{22}(z,\bar{k})}$. By a similar argument, $m_{21}(z,k) = \overline{m_{12}(z,\bar{k})}$. □

From Lemma 20, we can show a symmetry property for the off-diagonal entries of the scattering transform $S_\gamma(k)$.

Corollary 21. *Let $\gamma(z) = \sigma(z) + i\epsilon(z)$, where σ and ϵ satisfy* (3.69) *and* (3.70). *Let $(S_\gamma)_{12}(k)$ and $(S_\gamma)_{21}(k)$ be the off-diagonal entries of the scattering transform $S_\gamma(k)$. Then*

$$\overline{(S_\gamma)_{12}(k)} = (S_\gamma)_{21}(\bar{k}).$$

Proof. Let $(S_\gamma)_{12}(k)$ and $(S_\gamma)_{21}(k)$ be the off-diagonal entries of the scattering transform

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{C}} \begin{pmatrix} 0 & q(z)e_{-\bar{k}}(z)m_{22}(z,k) \\ -\bar{q}(z)e_k(z)m_{11}(z,k) & 0 \end{pmatrix} d\mu(z).$$

Then it follows by Lemma (20)

$$\begin{aligned} \overline{(S_\gamma)_{12}(k)} &= \frac{-i}{\pi}\int_{\mathbf{C}} \bar{q}(z)\overline{e_{-\bar{k}}(z)m_{22}(z,k)}\, d\mu(z), \\ &= \frac{-i}{\pi}\int_{\mathbf{C}} \bar{q}(z)e_{\bar{k}}(z)m_{11}(z,\bar{k})\, d\mu(z), \\ &= (S_\gamma)_{21}(\bar{k}). \end{aligned} \tag{5.23}$$

□

We will next prove that if $\gamma \in W^{1,p}(\Omega)$, then radial symmetry in the admittivity $\gamma \in W^{1,p}(\Omega)$ implies symmetry in the scattering transform S_γ. We will first need a simple lemma.

Lemma 22. *Let $\gamma(z) = \sigma(z) + i\omega\epsilon(z) \in W^{1,p}(\Omega)$, where σ and ϵ satisfy* (3.69) *and* (3.70). *Then $(\bar{\partial}\psi_{11})(z,k) = q(x)\psi_{21}(z,k)$ and $(\partial\psi_{21})(z,k) = \bar{q}(z)\psi_{11}(z,k)$.*

Proof. We know from (5.6) the d-bar relation $(\partial + ik)m_{21}(z,k) = \bar{q}(x)m_{11}(z,k)$ and from (5.9) $(\partial + ik)m_{21}(z,k) = e_{-k}(z)\partial(e_z(x)m_{21}(z,k))$. Thus,

$$\partial(e_k(z)m_{21}(z,k)) = e_k(z)\bar{q}(z)m_{11}(z,k). \tag{5.24}$$

It follows from (5.24) and (3.76) that

$$\partial(e_k(z)e^{-izk}\psi_{21}(z,k)) = e_k(z)\bar{q}(x)\psi_{11}(z,k)e^{-izk}. \tag{5.25}$$

Equivalently, we rewrite (5.25) as

$$\partial(e^{i\bar{z}\bar{k}}\psi_{21}(z,k)) = e^{i\bar{z}\bar{k}}\bar{q}(z)\psi_{11}(z,k). \tag{5.26}$$

Since

$$\partial(e^{i\bar{z}\bar{k}}\psi_{21}(z,k)) = (\partial(e^{i\bar{z}\bar{k}}))\psi_{21}(z,k) + e^{i\bar{z}\bar{k}}(\partial\psi_{21})(z,k) \tag{5.27}$$

and $\partial e^{i\bar{k}\bar{z}} = 0$, we get the following

$$(\partial\psi_{21})(z,k) = \bar{q}(z)\psi_{11}(z,k) \tag{5.28}$$

Similarly, $(\bar{\partial}\psi_{11})(z,k) = q(z)\psi_{21}(z,k)$. □

The following proposition is motivated by Knudsen [Knu02]

Proposition 23. *Let $\gamma(z) = \sigma(z) + i\omega\epsilon(z) \in W^{1,p}(\Omega)$, with $p > 2$ and let σ and ϵ satisfy (3.69) and (3.70). If $\gamma(z) = \gamma(e^{i\theta}z)$ for some angle θ. Then*

$$S_{21}(e^{i\theta}k) = e^{-i\theta}S_{21}(k) \text{ and } S_{12}(e^{i\theta}k) = e^{i\theta}S_{12}(k). \tag{5.29}$$

Proof. Since $\gamma(z) = \gamma(e^{i\theta}z)$, we have

$$q(z) = -\frac{\partial\gamma^{1/2}(z)}{\gamma^{1/2}(z)} = -\frac{\partial\gamma^{1/2}(e^{i\theta}z)}{\gamma^{1/2}(e^{i\theta}z)} = e^{i\theta}q(e^{i\theta}z). \tag{5.30}$$

Note that

$$\begin{aligned}\bar{\partial}\psi_{11}(e^{i\theta}z,k) =& e^{-i\theta}(\bar{\partial}\psi_{11})(e^{i\theta}z,k),\\ &= e^{-i\theta}q(e^{i\theta}z)\psi_{21}(e^{i\theta}z,k),\\ &= e^{-2i\theta}q(z)\psi_{21}(e^{i\theta}z,k)\end{aligned} \tag{5.31}$$

The three previous equalities came from the Chain Rule, Lemma 22, and (5.30), respectively. It is easy to see that $\psi_{11}(z, e^{i\theta}k)$ satisties the previous d-bar equation.

By similar reasonings, we get

$$\begin{aligned} \partial(e^{-2i\theta}\psi_{21}(e^{i\theta}z, k)) =& e^{-i\theta}(\partial\psi_{21})(e^{i\theta}z, k), \\ &= e^{-i\theta}\bar{q}(e^{i\theta}z)\psi_{11}(e^{i\theta}z, k), \\ &= \bar{q}(z)\psi_{11}(e^{i\theta}z, k) \end{aligned} \tag{5.32}$$

Observe that $\psi_{21}(z, e^{i\theta}k)$ also satisties the previous d-bar equation.

Thus we may conclude from Theorem (12) that

$$m_{11}(e^{i\theta}z, k) = m_{11}(z, e^{i\theta}k) \quad \textit{and} \quad m_{21}(e^{i\theta}z, k) = e^{2i\theta}m_{21}(z, e^{i\theta}k) \tag{5.33}$$

Thus, by (5.30), (5.33) and making a change of variable, we get

$$\begin{aligned} e^{i\theta}S_{21}(e^{i\theta}\ k) &= \frac{-i}{\pi}\int_{\mathbf{C}} e^{i\theta}\bar{q}(z)e(z, e^{i\theta}k)m_{11}(z, e^{i\theta}k)\, d\mu(z) \\ &= \frac{-i}{\pi}\int_{\mathbf{C}} \overline{e^{-i\theta}q(z)}e(z, e^{i\theta}k)m_{11}(z, e^{i\theta}k)\, d\mu(z) \\ &= \frac{-i}{\pi}\int_{\mathbf{C}} \overline{q(e^{i\theta}z)}e(e^{i\theta}z, k)m_{11}(z, e^{i\theta}k)\, d\mu(z) \\ &= \frac{-i}{\pi}\int_{\mathbf{C}} \bar{q}(w)e(w, k)m_{11}(w, k)\, d\mu(w) \\ &= S_{21}(k) \end{aligned} \tag{5.34}$$

By a similar argument, $S_{12}(e^{i\theta}k) = e^{i\theta}S_{12}(k)$ $\square$

Chapter 6

SOME PROPERTIES OF THE RECONSTRUCTION ALGORITHM

In this chapter, we will discuss some new results that we believe that will lead to an algorithm for reconstructing $\gamma = \sigma + i\omega\epsilon \in W^{1,p}(\Omega)$, $p > 2$ from the Dirichlet-to-Neumann map. Throughout we will be using Francini's definition of Q_γ and we will set $q = -\frac{1}{2}\partial \log \gamma$ and $\widetilde{q} = -\frac{1}{2}\bar{\partial} \log \gamma$.

We will show the existence of exponentially growing solutions (complex geometrical optics solutions) to the admittivity equation (1.1), but we will also show such solutions are unique under certain conditions. We have already that these central objects play a major role in the reconstruction algorithm of $\sigma \in W^{1,p}(\Omega)$, see section 3.2. We will see in this chapter that there are properties involving the exponentially growing solutions to the admittivity equation with $\gamma \in W^{1,p}(\Omega)$. In particular, we show that these exponentially growing solutions satisfy a certain boundary integral equation similar to (3.8). We will also discuss briefly some ideas towards the construction a boundary integral equation that involves off-diagonal entries of the scattering transform $S_\gamma(k)$ and the Dirichlet-to-Neumann map.

The properties that we will list in the next two sections should play a key role in the reconstruction algorithm from the Dirichlet-to-Neumann map of $\gamma \in W^{1,p}(\Omega)$, $p > 2$. The reconstruction algorithm proposed here for the admittivity $\gamma \in W^{1,p}(\Omega)$ consists of the following steps:

(1) Compute the trace on $\partial\Omega$ of the exponentially growing solutions to the admittivity equation from the boundary data.

(2) Compute the scattering transform.

(3) Solve a $\overline{\partial}$ equation, in the variable k for the exponentially growing solutions.

(4) Reconstruct admittivity using an appropriate equation that relates Q_γ to the exponentially growing solutions to the admittivity equation.

Francini's work provides the $\bar{\partial}$ equation needed in step (3) and an equation relating Q_γ to the exponentially growing solutions, needed for step (4). The definition of the scattering transform requires knowledge of S_γ, which is not possible in the inverse problem. In section 6.3, we will derive an equation relating the scattering transform to the boundary data through the trace to the boundary of the exponentially growing solutions ψ. In turn we require knowledge of the trace of ψ on $\partial\Omega$, and we derive an equation for that as well in section 6.2.

The chapter is outline as follows. In section 6.1, we will discuss some properties of the exponentially growing solutions to the admittivity equation that will be useful in later section for the reconstruction algorithm of γ. In section 6.2, we derive an important boundary integral equation that is very similar (3.8) and is a key step in the reconstruction algorithm. In section 6.3, we discuss some ideas that we think will lead to a boundary integral equation similar to (3.43) in the near future. In section 6.4, we will briefly mention how can we proceed with steps (3) and (4) using the work of Francini.

6.1 Uniqueness of a Certain Exponentially Growing Solution

We will need first need some results of Nachman and Francini before we establish a connection of the uniqueness of a certain exponentially growing solution to the admittivity equation (1.1) analogous to Theorem 6. This result will be used later in this section for establishing a boundary integral equation involving a certain exponentially growing solution to the admittivity equation.

Theorem 24. *(part of Theorem 4.1 of [Fra00]) Let $\gamma(z) = \sigma(z) + i\epsilon(z)$, where σ and ϵ satisfy* (3.69) *and* (3.70), *respectively. Then there exists a $K > 0$ such that for all $z \in \mathbf{C}$, $\sup \|M(z,\cdot) - I\|_{L^p(\mathbf{R}^2)} \leq K$ for every $p > 2$, and K depends on the constants E, σ_0 and p.*

The following Lemma in the real sense was first used by Nachman, the complex version also holds and was used in [Fra00].

Lemma 25. *(Nachman [Nac96]) Let $1 < s < 2$ and $\frac{1}{r} = \frac{1}{s} - \frac{1}{2}$.*

(1) If the complex function $v \in L^s(\mathbf{R}^2)$, then there exists a unique complex function $u \in L^r(\mathbf{R}^2)$ such that $(\partial + ik)u = v$.

(2) If the complex function $v \in L^r(\mathbf{R}^2)$ and $\bar{\partial} v \in L^s(\mathbf{R}^2)$, $k \in \mathbf{C} - \{0\}$, then there exists a unique complex function $u \in W^{1,r}(\mathbf{R}^2)$ such that $(\partial + ik)u = v$.

(3) If the complex function $v \in L^r(\mathbf{R}^2)$ and $\bar{\partial} v \in L^s(\mathbf{R}^2)$, $k \in \mathbf{C} - \{0\}$, then there exists a unique complex function $u \in W^{1,r}(\mathbf{R}^2)$ such that $(\bar{\partial} - ik)u = v$.

The above lemma is also true if $\bar{\partial}$ is replaced by ∂.

The following lemma will also be used in the proof of relating exponentially growing solutions to the admittivity equation (1.1).

Lemma 26. *Let $\gamma(z) \in W^{1,p}(z,k)$. Then we have the following identities*

$$\bar{\partial}(\gamma(z)^{-1/2}m_{11}(z,k) - 1) = (\partial + ik)(\gamma(z)^{-1/2}m_{21}(z,k)) \tag{6.1}$$

$$\partial(\gamma(z)^{-1/2}m_{22}(z,k) - 1) = (\bar{\partial} - ik)(\gamma(z)^{-1/2}m_{12}(z,k)) \tag{6.2}$$

Proof. Note that

$$\begin{aligned}\bar{\partial}(\gamma(z)^{-1/2}m_{11}(z,k) - 1) &= \bar{\partial}(\gamma(z)^{-1/2})m_{11}(z,k) + \gamma(z)^{-1/2}\bar{\partial}(m_{11}(z,k)) \\ &= \gamma(z)^{-1/2}\widetilde{q}(z)m_{11}(z,k) + \gamma(z)^{-1/2}q(z)m_{21}(z,k) \\ &= \gamma(z)^{-1/2}(\partial + ik)m_{21}(z,k) + \gamma(z)^{-1/2}q(z)m_{21}(z,k)\end{aligned} \tag{6.3}$$

The second and third equalities came from (3.79) and (3.80), respectively.

We also have

$$\begin{aligned}(\partial + ik)(\gamma(z)^{-1/2}m_{21}(z,k)) &= \partial(\gamma(z)^{-1/2}m_{21}(z,k)) + ik\gamma(z)^{-1/2}m_{21}(z,k) \\ &= \partial(\gamma(z)^{-1/2})m_{21}(z,k) + \gamma(z)^{-1/2}\partial(m_{21}(z,k)) \\ &+ ik\gamma(z)^{-1/2}m_{21}(z,k) \\ &= \gamma(z)^{-1/2}q(z)m_{21}(z,k) + \gamma(z)^{-1/2}(\partial + ik)m_{21}(z,k)\end{aligned} \tag{6.4}$$

Thus we get (6.1).

We will now derive (6.2). From (3.81) and (3.82), we get

$$\begin{aligned}\partial(\gamma(z)^{-1/2}m_{22}(z,k) - 1) &= \partial(\gamma(z)^{-1/2})m_{22}(z,k) + \gamma(z)^{-1/2}\partial(m_{22}(z,k)) \\ &= \gamma(z)^{-1/2}q(z)m_{22}(z,k) + \gamma(z)^{-1/2}\widetilde{q}(z)m_{12}(z,k) \\ &= \gamma(z)^{-1/2}(\bar{\partial} - ik)m_{12}(z,k) + \gamma(z)^{-1/2}\widetilde{q}(z)m_{12}(z,k)\end{aligned} \tag{6.5}$$

We also have

$$
\begin{aligned}
(\bar{\partial} - ik)(\gamma(z)^{-1/2}m_{12}(z,k)) &= \bar{\partial}(\gamma(z)^{-1/2}m_{12}(z,k)) - ik\gamma(z)^{-1/2}m_{12}(z,k) \\
&= \bar{\partial}(\gamma(z)^{-1/2})m_{12}(z,k) + \gamma(z)^{-1/2}\bar{\partial}(m_{12}(z,k)) \\
&\quad - ik\gamma(z)^{-1/2}m_{12}(z,k) \\
&= \gamma(z)^{-1/2}\widetilde{q}(z)m_{12}(z,k) + \gamma(z)^{-1/2}(\bar{\partial} - ik)m_{12}(z,k)
\end{aligned}
\tag{6.6}
$$

Thus we get (6.2). □

If $\sigma \in W^{1,p}(\Omega)$, we saw in section 3.2 that exponentially growing solutions to the conductivity equation (1.1) if $\gamma = \sigma$, played a major role in the proof of constructing a boundary integral equation that connect certain off-diagonal scattering transform of the scattering transform $S_\sigma(k)$ and the Dirichlet-to-Neumann map. We will use the idea of [BBR01] to prove some results for the existence of exponentially growing solutions to the admittivity equation for the complex-valued $\gamma \in W^{1,p}(\mathbf{R}^2)$.

Theorem 27. *Let $\gamma(z) = \sigma(z) + i\omega\epsilon(z) \in W^{1,p}(\Omega)$, with $p > 2$ and such that σ and ϵ satisfy (3.69) and (3.70), and let $\gamma(z) - 1$ have compact support in $W^{1,p}(\Omega)$. Then for all $k \in \mathbf{C} \setminus \{0\}$ there exists a unique solution $u(z,k) = e^{ikz}[\frac{1}{ik} + w(z,k)]$ to the admittivity equation $\nabla \cdot \gamma \nabla u = 0$ in $\mathbf{R}^2$ such that $w(\cdot, k) \in W^{1,r}(\mathbf{R}^2)$, $2 < r < \infty$. Moreover, we get the following equalities*

$$(\partial + ik)(e^{-ikz}u(z,k) - \frac{1}{ik}) = \gamma^{-1/2}(z)m_{11}(z,k) - 1 \tag{6.7}$$

$$\bar{\partial}(e^{-ikz}u(z,k) - \frac{1}{ik}) = \gamma^{-1/2}(z)m_{21}(z,k) \tag{6.8}$$

and

$$\left\| e^{-ikz}u(z,k) - \frac{1}{ik} \right\|_{W^{1,r}(\mathbf{R}^2)} \leq C(1 + \frac{1}{|k|}) \tag{6.9}$$

for some constant C.

Proof. Let γ be as given as in the statement of the theorem. Define the complex function v via $v(z) = \gamma(z)^{-1/2}m_{11}(z,k) - 1$. We will first show there exists a unique complex function $w \in W^{1,r}(\mathbf{R}^2)$, $r > 2$ such that $(\partial + ik)w = v$, $k \in \mathbf{C} - \{0\}$. Let us rewrite v as follows,

$$v(z) = \gamma(z)^{-1/2}[m_{11}(z,k) - 1] + [\gamma(z)^{-1/2} - 1].$$

Let $r > 2$ and $1 < s < 2$ with $\frac{1}{r} = \frac{1}{s} - \frac{1}{2}$. We know by Theorem 24 that there exists a $C > 0$ such that $sup\, \|m_{11}(z,k) - 1\|_{L^r(\mathbf{R}^2)} \leq C$ for every $r > 2$ and $\gamma(z)^{-1/2} - 1$ has compact support in $W^{1,r}(\mathbf{R}^2)$. It follows that $v \in L^r(\mathbf{R}^2)$ and by Minkowski's Inequality

$$\|v(z)\|_{L^r} = \left\|\gamma(z)^{-1/2}[m_{11}(z,k) - 1] + [\gamma(z)^{-1/2} - 1]\right\|_{L^r} \leq C_{r,\gamma},$$

where $C_{r,\gamma}$ depends on r and bounds on σ and ϵ.

We have

$$\begin{aligned}\bar{\partial}v(z) &= \bar{\partial}(\gamma(z)^{-1/2}m_{11}(z,k) - 1)\\ &= \bar{\partial}(\gamma(z)^{-1/2})m_{11}(z,k) + \gamma(z)^{-1/2}(\bar{\partial}m_{11}(z,k))\\ &= \gamma(z)^{-1/2}\widetilde{q}(z)m_{11}(z,k) + \gamma(z)^{-1/2}q(z)m_{21}(z,k)\\ &= \gamma(z)^{-1/2}\widetilde{q}(z)[m_{11}(z,k) - 1] + \gamma(z)^{-1/2}q(z)m_{21}(z,k) + \gamma(z)^{-1/2}q(z)\end{aligned} \tag{6.10}$$

The third equality came from (3.100).

We know that $\gamma(z)^{-1/2}\widetilde{q}(z) \in L^q(\mathbf{R}^2)$ with $1 \leq q \leq p$ because $q(z)$ has compact support. It follows that $\gamma(z)^{-1/2}\widetilde{q}(z) \in L^s(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$. By Generalized Hölder's Inequality and the fact that $\|m_{11}(z,k) - 1\|_{L^s}$ is bounded with $\frac{1}{s} = \frac{1}{r} + \frac{1}{2}$, we have $\bar{\partial}v(z) \in L^s(\mathbf{R}^2)$ and $\left\|\bar{\partial}v\right\|_{L^s(\mathbf{R}^2)} \leq K_{r,\gamma}$, where $K_{r,\gamma}$ depends only on r and bounds on σ and ϵ. Thus by Lemma 25 (2) there exists a unique solution $w(z,k) \in W^{1,r}(\mathbf{R}^2)$ such that

$$(\partial + ik)w(z,k) = \gamma(z)^{-1/2}m_{11}(z,k) - 1 \tag{6.11}$$

We have by (6.1),

$$\bar{\partial}(\gamma(z)^{-1/2}m_{11}(z,k) - 1) = \partial_k(\gamma(z)^{-1/2}m_{21}(z,k)) \tag{6.12}$$

Taking $\bar{\partial}$ of both sides of (6.11) and using (6.12), we get

$$\begin{aligned} \bar{\partial}(\partial + ik)w(z,k) &= \bar{\partial}(\gamma(z)^{-1/2}m_{11}(z,k) - 1) \\ &= (\partial + ik)(\gamma(z)^{-1/2}m_{21}(z,k)) \end{aligned} \tag{6.13}$$

Hence it follows by using the fact $\bar{\partial}(\partial + ik) = (\partial + ik)\bar{\partial}$,

$$(\partial + ik)(\bar{\partial}w(z,k) - \gamma(z)^{-1/2}m_{21}(z,k)) = 0 \tag{6.14}$$

But $\bar{\partial}w(z,k) - \gamma(z)^{-1/2}m_{21}(z,k) \in L^r(\mathbf{R}^2)$, and so Lemma 25 (1), we must have

$$\bar{\partial}w(z,k) = \gamma(z)^{-1/2}m_{21}(z,k). \tag{6.15}$$

Let's define

$$u(z,k) = e^{ikz}[w(z,k) + \frac{1}{ik}]. \tag{6.16}$$

Thus, by (6.11)

$$(\partial + ik)(e^{-ikz}u(x,k) - \frac{1}{ik}) = (\partial + ik)w(z,k) = \gamma^{-1/2}(z)m_{11}(z,k) - 1$$

and by (6.15)

$$\bar{\partial}(e^{-ikz}u(x,k) - \frac{1}{ik}) = \bar{\partial}w(z,k) = \gamma^{-1/2}(z)m_{21}(z,k)$$

That is, (6.16) satisfies (6.7) and (6.8).

The norm estimate given by (6.9) follows by Minkowski's Inequality, the constant C depends on r, the bound on $\gamma - 1$, and bounds on σ and ϵ. $\square$

Remark: Note that from (6.7) we get

$$\begin{aligned}\gamma^{-1/2}m_{11}(z,k)-1&=(\partial+ik)(e^{-ikz}u(z,k)-\frac{1}{ik})\\&=\partial(e^{-ikz}u)+ike^{-ikz}\partial u(z,k)\\&=e^{-ikz}\partial u(z,k)\end{aligned}\tag{6.17}$$

and from (6.8)

$$\begin{aligned}\gamma^{-1/2}m_{21}(z,k)&=\bar\partial(e^{-ikz}u(z,k)-\frac{1}{ik})\\&=u(z,k)\bar\partial(e^{-ikz})+e^{-ikz}\bar\partial u(z,k)\\&=e^{-ikz}\bar\partial u(z,k).\end{aligned}\tag{6.18}$$

Equivalently, we can rewrite (6.7) and (6.8), respectively, as

$$\gamma^{1/2}(z)\partial u(z,k)=e^{ikz}m_{11}(z,k)\tag{6.19}$$

$$\gamma^{1/2}(z)\bar\partial u(z,k)=e^{ikz}m_{21}(z,k).\tag{6.20}$$

We can also get an analogous result of Theorem 27 for exponentially growing solutions to the admittivity equation that are very similar to (6.7) and (6.8) involving $m_{22}(z,k)$ and $m_{12}(z,k)$.

Theorem 28. *Let $\gamma(z)=\sigma(z)+i\omega\epsilon(z)\in W^{1,p}(\Omega)$, with $p>2$ and such that σ and ϵ satisfy* (3.69) *and* (3.70), *and let $\gamma(z)-1$ have compact support in $W^{1,p}(\Omega)$. Then for all $k\in\mathbf{C}\setminus\{0\}$ there exists a unique solution $\widetilde{u}(z,k)=e^{ik\bar z}[\frac{1}{ik}+\widetilde{w}(z,k)]$ to the admittivity equation $\nabla\cdot\gamma\nabla\widetilde{u}=0$ in $\mathbf{R}^2$ with $\widetilde{w}(\cdot,k)\in W^{1,r}(\mathbf{R}^2)$, $2<r<\infty$. Moreover, we get the following equalities*

$$(\bar\partial-ik)(e^{-ik\bar z}\widetilde{u}(z,k)-\frac{1}{ik})=\gamma^{-1/2}(z)m_{22}(z,k)-1\tag{6.21}$$

$$\partial(e^{-ik\bar{z}}\widetilde{u}(z,k) - \frac{1}{ik}) = \gamma^{-1/2}(z)m_{12}(z,k). \tag{6.22}$$

and

$$\left\| e^{-ik\bar{z}}\widetilde{u}(z,k) - \frac{1}{ik} \right\|_{W^{1,r}(\mathbf{R}^2)} \leq C(1 + \frac{1}{|k|}) \tag{6.23}$$

for some constant C.

Proof. The proof is identical to Theorem 27 and so we will only sketch some of the ideas we need to modify the proof. Let γ be as given as in the statement of the theorem. Define the complex function v via $v = \gamma(z)^{-1/2}m_{22}(z,k) - 1$ One can show that $v \in L^r(\mathbf{R}^2)$ and $\partial v \in L^s(\mathbf{R}^2)$, where $r > 2$ and $1 < s < 2$ with $\frac{1}{r} = \frac{1}{s} - \frac{1}{2}$.

By Lemma 25 (3) there exists a unique complex function $\tilde{w} \in W^{1,r}(\mathbf{R}^2)$, such that

$$(\bar{\partial} - ik)\widetilde{w}(z,k) = \gamma(z)^{-1/2}m_{22}(z,k) - 1 \tag{6.24}$$

By (6.2), we have

$$\partial(\gamma(z)^{-1/2}m_{22}(z,k) - 1) = (\bar{\partial} - ik)(\gamma(z)^{-1/2}m_{12}(z,k)) \tag{6.25}$$

Taking ∂ of both sides of (6.24), the fact that $\partial(\bar{\partial} - ik) = (\bar{\partial} - ik)\partial$ and using 6.25, we get

$$(\bar{\partial} - ik)(\partial\widetilde{w}(z,k) - \gamma(z)^{-1/2}m_{12}(z,k)) = 0 \tag{6.26}$$

But $\partial\widetilde{w}(z,k) - \gamma(z)^{-1/2}m_{12}(z,k) \in L^r(\mathbf{R}^2)$, and so by Lemma 25 (1), we must have

$$\partial\widetilde{w}(z,k) = \gamma(z)^{-1/2}m_{12}(z,k). \tag{6.27}$$

Let's define

$$\widetilde{u}(z,k) = e^{ik\bar{z}}[\frac{1}{ik} + \widetilde{w}(z,k)]. \tag{6.28}$$

Then the equations (6.21) and (6.22) follow with $W^{1,r}(\mathbf{R}^2)$ norm given by (6.23). $\square$

Equivalently, we can rewrite (6.21) and (6.22), respectively, as

$$\gamma^{1/2}(z)\bar{\partial}\widetilde{u}(z,k) = e^{ik\bar{z}}m_{22}(z,k) \tag{6.29}$$

$$\gamma^{1/2}(z)\partial\widetilde{u}(z,k) = e^{ikz}m_{12}(z,k). \tag{6.30}$$

6.2 A Boundary Integral Equation Involving a Exponentially Growing Solution to The Admittivity Equation

In this section, we will show that the exponentially growing solutions u to the admittivity equation as given in Theorem 27 satisfy a boundary integral equation that is similar to both (3.8) and (3.63).

We will begin this section by establishing a simple but useful integral identity.

Lemma 29. *Let $\gamma_1, \gamma_2 \in L^\infty(\Omega)$. If $u_1, u_2 \in H^1(\Omega)$ satisfy the admittivity equation (1.1) with boundary values f_i, $i = 1, 2$ then*

$$\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2})f_1, f_2\rangle = \int_\Omega (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla u_2 dx. \tag{6.31}$$

Proof. We know that the Dirichlet-to-Neumann map Λ_γ is a self-adjoint operator. Thus, we get

$$\begin{aligned}
\langle(\Lambda_{\gamma_1} - \Lambda_{\gamma_2} f_1, f_2\rangle &= \langle\Lambda_{\gamma_1} f_1, f_2\rangle - \langle\Lambda_{\gamma_2} f_1, f_2\rangle \\
&= \langle\Lambda_{\gamma_1} f_1, f_2\rangle - \overline{\langle\Lambda_{\gamma_2} f_2, f_1\rangle} \\
&= \int_\Omega \gamma_1 \nabla u_1 \cdot \nabla \bar{u}_2 dx - \int_\Omega \gamma_2 \nabla \bar{u}_2 \cdot \nabla u_1 dx \\
&= \int_\Omega (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla \bar{u}_2 dx.
\end{aligned} \tag{6.32}$$

$\square$

This integral identity implies the map $\Lambda : \gamma \mapsto \Lambda_\gamma$ is continuous with respect to γ.

It was claimed by Bukhgeim in [Buk07] that the results of [Buk07] will yield global uniqueness for the Calderón problem with a complex-valued conductivity (i.e., admittivity). So we will assume Lemma 2 holds for complex measurable function $\gamma \in W^{1,p}(\Omega)$, where $1 < p < 2$. There is also numerical evidence that supports this claim from computations performed by E. Murphy. The following proposition shows a relationship between exponentially growing solutions $\psi(z,k)$ to the Schrödinger equation and $u(z,k)$ to the admittivity equation. We will also use the fact that by the Sobolev Embedding Theorem, $W^{2,p}(\Omega) \subset W^{1,p}(\Omega)$, with $1 < p < 2$.

Proposition 30. *Let $\gamma(z) = \sigma(z) + i\omega\epsilon(z) \in W^{1,p}(\Omega)$, with $1 < p < 2$. Suppose that $\gamma = 1$ near $\partial\Omega$, σ and ϵ satisfy* (3.69) *and* (3.70), *and let $\gamma(z) - 1$ have compact support in $W^{1,p}(\Omega)$. Let u be the exponentially growing solution to the admittivity equation as given in Theorem 27 and let ψ be the exponentially growing solution to the Schrödinger equation. Then for nonexceptional $k \in \boldsymbol{C} - \{0\}$*

$$iku(z,k) = \gamma^{-1/2}(z)\psi(z,k) \tag{6.33}$$

Proof. Note that

$$\begin{aligned} iku(z,k) &= e^{ikz}(1 + ikw(z,k)) \\ &= e^{ikz}\gamma(z)^{-1/2}(\gamma(z)^{1/2} + \gamma(z)^{1/2}ikw(z,k)) \\ &= e^{ikz}\gamma(z)^{-1/2}(1 + (\gamma(z)^{1/2} - 1) + \gamma(z)^{1/2}ikw(z,k)) \end{aligned} \tag{6.34}$$

satisfies the admittivity equation with

$$\gamma(z)^{1/2} - 1 + \gamma(z)^{1/2} ikw(z,k) \in W^{1,r}(\Omega)$$

with $r > 2$.

Let $\psi(z,k)$ be the exponential growing solution to the Schrödinger equation,

$$(-\Delta + q)\psi(z,k) = 0, \;\; z \in \mathbf{C} \quad \mathrm{k} \in \mathbf{C} - \{0\}.$$

Here $q = \frac{\Delta\gamma^{1/2}}{\gamma^{1/2}}$.

We also know that

$$\gamma(z)^{-1/2}\psi(z,k) = e^{ikz}\gamma(z)^{-1/2}(1 + \tilde{w}(z,k))$$

is also a solution to the admittivity equation with $\tilde{w}(z,k) \in W^{1,\tilde{p}}(\mathbf{R}^2)$. Hence by Theorem 27 and the complex version of Lemma 2, these exponentially growing solutions must be equal. $\square$

We will recall some terminology before we establish a boundary integral equation involving exponentially growing solutions. The single layer potential S_k is a boundary integral operator defined by

$$S_k f(z) = \int_{\partial\Omega} f(y) G_k(z-y) d\mu(y). \tag{6.35}$$

where G_k is the Faddeev Green's function (3.6).

Now we are ready to establish an important boundary integral equation.

Theorem 31. *Let $\gamma \in W^{1,p}(\Omega)$ for $p > 2$ and suppose $\gamma = 1$ near $\partial\Omega$. Suppose σ and ϵ satisfy* (3.69) *and* (3.70), *and let $\gamma(z) - 1$ have compact support in $W^{1,p}(\Omega)$. Then for any nonexceptional $k \in \boldsymbol{C} - \{0\}$ the trace of the exponentially growing solution $u(\cdot,k)$ on $\partial\Omega$ is the unique solution to*

$$u(z,k) = \frac{1}{ik} e^{ikz} - S_k(\Lambda_\gamma - \Lambda_1)u(\cdot,k), \quad z \in \partial\Omega \tag{6.36}$$

$k \in \mathbf{C} - \{0\}$, Λ_γ *is the voltage-to-current map when* Ω *contains the admittivity distribution and* Λ_1 *is the Dirichlet-to-Neumann map of the homogeneous admittivity 1.*

Proof. Let $\frac{1}{p} = \frac{1}{r} - \frac{1}{2}$, where $1 < r < 2$ and $p > 2$. Let $\{\gamma_n\}_{n\in\mathbf{N}} \subset W^{2,r}(\Omega)$ be a sequence converging to $\gamma \in W^{1,p}(\Omega)$ and so by the Sobolev Embedding Theorem, $\{\gamma_n\}_{n\in\mathbf{N}} \subset W^{1,r}(\Omega)$. Let ψ_n be the exponentially growing solutions to the Schrödinger equation with potential $\gamma_n^{-1/2}\Delta\gamma_n^{1/2}$, and u_n be the exponentially growing solutions to the admittivity equation with admittivity γ_n. So we know for each $n \in \mathbf{N}$, the complex version of (3.8) holds for nonexceptional $k \in \mathbf{C} - \{0\}$

$$\psi(\cdot, k)|_{\partial\Omega} = e^{ikz}|_{\partial\Omega} - S_k(\Lambda_\gamma - \Lambda_1)\psi(\cdot, k),$$

$k \in \mathbf{C} - \{0\}$, and $\gamma = 1$ in the neighborhood of $\partial\Omega$.

It follows by (6.33) that for each complex number $k \neq 0$, and for each $n \in \mathbf{N}$

$$\frac{\gamma_n{}^{-1/2}}{ik}\psi_n = u_n(\cdot, k) \to u(\cdot, k) \quad \text{in} \quad H^{1/2}(\partial\Omega). \tag{6.37}$$

We claim that for each n, u_n satisfies (6.36). To see this, by (6.33) and $\gamma_n = 1$ in a neighborhood of $\partial\Omega$,

$$\begin{aligned}
\frac{1}{ik}e^{ikz}|_{\partial\Omega} - S_k(\Lambda_\gamma - \Lambda_1)u_n(\cdot, k) &= \frac{1}{ik}e^{ikz}|_{\partial\Omega} - S_k(\Lambda_\gamma - \Lambda_1)\frac{\gamma_n{}^{-1/2}}{ik}\psi_n(\cdot, k) \\
&= \frac{1}{ik}e^{ikz}|_{\partial\Omega} - S_k(\Lambda_\gamma - \Lambda_1)\frac{1}{ik}\psi_n(\cdot, k) \\
&= \frac{\psi_n(z, k)}{ik}|_{\partial\Omega} \\
&= \frac{\gamma_n^{-1/2}}{ik}\psi_n(z, k)|_{\partial\Omega} \\
&= u_n(z, k).
\end{aligned} \tag{6.38}$$

Thus, u_n satisfies (6.36) for each $n \in \mathbf{N}$.

We know by Theorem 12 that $M(z,k)$ depends continuously on γ. From (6.37) and (6.31), we can conclude that

$$S_k(\Lambda_{\gamma_n} - \Lambda_1)u_n(\cdot,k) \mapsto S_k(\Lambda_\gamma - \Lambda_1)u(\cdot,k) \tag{6.39}$$

So from (6.38), (6.37) and (6.39), $u(\cdot,k)|_{\partial\Omega}$ satisfies (6.36). The uniqueness of $u(\cdot,k)|_{\partial\Omega}$ follows by Theorem 27. $\square$

6.3 The Scattering Transform and The D-N Map

Let us recall the scattering transform $S_\gamma(k)$ from section 3.3,

$$S_\gamma(k) = \frac{i}{\pi}\int_{\mathbf{R}^2}\begin{pmatrix} 0 & e^{-i\bar{k}z}q(z)\psi_{22}(z,k) \\ -e^{ik\bar{z}}\overline{q}(z)\psi_{11}(z,k) & 0 \end{pmatrix} d\mu(z).$$

Note that by using the fact that $\bar{\partial}(e^{-i\bar{k}z}\psi_{12}) = e^{-i\bar{k}z}q\psi_{22}$, integration by parts, and by the compact support of q the off-diagonal entry $(S_\gamma)_{12}(k)$ of $S_\gamma(k)$ can be written as

$$\begin{aligned}(S_\gamma)_{12}(k) &= \frac{i}{\pi}\int_{\mathbf{R}^2} e^{-i\bar{k}z}q(z)\psi_{22}(z,k)d\mu(z) \\ &= \frac{i}{\pi}\int_{\Omega} e^{-i\bar{k}z}q(z)\psi_{22}(z,k)d\mu(z) \\ &= \frac{i}{\pi}\int_{\Omega} \bar{\partial}(e^{-i\bar{k}z}\psi_{12}(z,k))d\mu(z) \\ &= \frac{i}{2\pi}\int_{\partial\Omega} \bar{\nu}e^{-i\bar{k}z}\psi_{12}(z,k)d\mu(z)\end{aligned} \tag{6.40}$$

where $\nu = v_1 + iv_2$ denotes the unit outer normal vector to $\partial\Omega$. By similar argument, we get a formula for the other off-diagonal entry of $S_\gamma(k)$ given by

$$(S_\gamma)_{21}(k) = \frac{i}{2\pi}\int_{\partial\Omega} \nu e^{-i\bar{k}z}\psi_{21}(z,k)d\mu(z). \tag{6.41}$$

We have seen that the reconstruction method uses the reduction of the conductivity equation to a first order elliptic system, and applying the d-bar method of inverse scattering theory to this elliptic system. In [BBR01], the authors established a boundary integral equation that connects the off-diagonal entries of the scattering transform $S_\sigma(k)$ and the Dirichlet-to-Neumann map. We will continue to use the idea of [BBR01] to support that the scattering transform $S_\gamma(k)$ can be found by using the exponentially growing solutions $u(z,k)$ to the admittivity equation. The authors generalized a certain formula which was first established by Alessandrini [Ale90], and they also used some of the ideas of [Liu97]. We have already seen that several properties of the exponentially growing solutions u to the conductivity equation in Theorem 6 are some of the key ingredients in the proof of Theorem 7. Moreover, the symmetry (3.30) was used in the proof of Theorem 7.

We would like to prove an analogous result of Theorem 7, but there are some problems with coming up with such a result. We don't have any symmetries in the entries of the Jost matrix M with Francini's definition of the potential matrix Q_γ. We also know my definition of the potential matrix Q_γ will not work for several reasons, one reason is that we won't have a version of Theorem 27. In this section we will discuss a possible approach to establish a boundary integral equation that connects the off-diagonal entries of the scattering transform $S_\gamma(k)$ and the Dirichlet-to-Neumann map.

We let Q_i be the potential matrices associated to the admittivities γ_i,

$$Q_i = \begin{pmatrix} 0 & q_i \\ \widetilde{q}_i & 0 \end{pmatrix}$$

where $q_i = -\frac{1}{2}\partial \log \gamma_i$ and $\widetilde{q}_i = -\frac{1}{2}\bar{\partial} \log \gamma_i$. We let $M(Q_i, x, k)$ be the corresponding Jost matrices.

Let's recall that the exponentially growing solutions u from Theorems 27 and 28, respectively, satisfy

$$\gamma_2^{1/2}(z)\partial u(Q_2, z, k) = e^{ikz} m_{11}(Q_2, z, k) \tag{6.42}$$

and

$$\gamma_1^{1/2}(z)\bar{\partial} u(Q_1, z, k) = e^{ik\bar{z}} m_{22}(Q_1, z, k) \tag{6.43}$$

After taking $(\partial + ik)^{-1}$ of both sides of (6.7), we set

$$R(Q_2, z, k) = e^{-ikz} u(Q_2, z, k) - \frac{1}{ik}. \tag{6.44}$$

From (6.21), we can define $\widetilde{R}$ in a similar manner as above,

$$\widetilde{R}(Q_1, z, k) = e^{ik\bar{z}} u(Q_1, z, k) - \frac{1}{ik}. \tag{6.45}$$

Define

$$\begin{aligned} I = &\int_\Omega \bar{\partial}(u(Q_1, z, k)u(Q_2, z, -\bar{k}))\gamma_2{}^{1/2}\partial\gamma_1{}^{1/2}dz \\ &- \int_\Omega \partial(u(Q_1, z, -\bar{k})u(Q_2, z, k))\gamma_1{}^{1/2}\bar{\partial}\gamma_2{}^{1/2}dz. \end{aligned} \tag{6.46}$$

We will prove that:

$$I = -\frac{\pi}{k}(S_{\gamma_1})_{21}(k) + \frac{\pi}{\bar{k}}(S_{\gamma_2})_{21}(k).$$

The first integral of I, I_1, simplifies to

$$\begin{aligned} I_1 = &- e_{-\bar{k}}(z) m_{22}(Q_1, z, k)\gamma_2{}^{1/2} q_1 (R(Q_2, z, -\bar{k}) - \frac{1}{i\bar{k}}) \\ &- e_{-\bar{k}}(z) m_{21}(Q_2, z, -\bar{k})\gamma_1{}^{1/2} q_1 (R(Q_1, z, k) + \frac{1}{ik}). \end{aligned} \tag{6.47}$$

The second integral of I, I_2, simplifies to

$$\begin{aligned} I_2 = &- e_k(z) m_{12}(Q_1, z, -\bar{k})\gamma_2{}^{1/2} \widetilde{q}_2 (R(Q_2, z, k) + \frac{1}{ik}) \\ &- e_k(z) m_{11}(Q_1, z, k)\gamma_1{}^{1/2} \widetilde{q}_2 (R(Q_1, z, -\bar{k}) - \frac{1}{i\bar{k}}). \end{aligned} \tag{6.48}$$

From (6.47), (6.48), we can set $I = J_1 + J_2$, where

$$\begin{aligned}
J_1 = &- \frac{1}{i\bar{k}} \int_\Omega e_{-\bar{k}}(z) m_{22}(Q_1, z, k) q_1(z) {\gamma_2}^{1/2}(z) dz \\
&- \frac{1}{ik} \int_\Omega e_{-\bar{k}}(z) m_{21}(Q_2, z, -\bar{k}) q_1(z) {\gamma_1}^{1/2}(z) dz \\
&+ \frac{1}{ik} \int_\Omega e_k(z) m_{12}(Q_1, x, -\bar{k}) \widetilde{q}_2(z) {\gamma_2}^{1/2}(z) dz \\
&- \frac{1}{i\bar{k}} \int_\Omega e_k(z) m_{11}(Q_2, z, k) \widetilde{q}_2(z) {\gamma_1}^{1/2}(x) dz
\end{aligned} \tag{6.49}$$

and

$$\begin{aligned}
J_2 = &- \int_\Omega e_{-\bar{k}}(z) m_{22}(Q_1, x, k) q_1(x) {\gamma_2}^{1/2}(z) R(Q_2, z, -\bar{k}) dz \\
&- \int_\Omega e_{-\bar{k}}(z) m_{21}(Q_2, z, -\bar{k}) q_1(z) {\gamma_1}^{1/2}(z) R(Q_1, z, k) dz \\
&+ \int_\Omega e_k(z) m_{12}(Q_1, z, -\bar{k}) \widetilde{q}_2(z) {\gamma_2}^{1/2}(z) R(Q_2, z, k) dz \\
&+ \int_\Omega e_k(z) m_{11}(Q_2, z, k) \widetilde{q}_2(z) {\gamma_1}^{1/2}(z) R(Q_1, z, -\bar{k}) dz
\end{aligned} \tag{6.50}$$

Notice

$$\begin{aligned}
J_1 = &- \frac{\pi}{\bar{k}} (S_{\gamma_1})_{12}(k) - \frac{\pi}{k} (S_{\gamma_2})_{21}(k) \\
&+ \frac{1}{ik} \int_\Omega e_{-\bar{k}}(z) m_{22}(Q_1, z, k) q_1(z) ({\gamma_2}^{1/2}(z) - 1) dz \\
&- \frac{1}{ik} \int_\Omega e_{-\bar{k}}(z) m_{21}(Q_1, z, -\bar{k}) q_1(z) {\gamma_1}^{1/2}(z) dz \\
&+ \frac{1}{ik} \int_\Omega e_k(z) m_{12}(Q_1, z, -\bar{k}) \widetilde{q}_2(z) {\gamma_1}^{1/2}(z) dz \\
&- \frac{1}{i\bar{k}} \int_\Omega e_k(z) m_{11}(Q_2, z, k) \widetilde{q}_2(z) ({\gamma_1}^{1/2}(z) - 1) dz.
\end{aligned} \tag{6.51}$$

So (6.51) is starting to look like (3.50) except we are not using any symmetries of the entries of the Jost matrix M.

We also have from (3.80), (3.83), (3.81) and (3.84),

$$e_{-\bar{k}}(z) m_{22}(Q_1, z, k) q_1(z) = \bar{\partial}(e_{-\bar{k}}(z) m_{12}(Q_1, z, k)) \tag{6.52}$$

$$e_k(z) m_{11}(Q_2, z, k) \widetilde{q}_2(z) = \partial(e_k(z) m_{21}(Q_2, z, k)), \tag{6.53}$$

Substituting (6.52) and (6.53) back into (6.51), then use integration by parts, and the fact $\gamma_i = 1$ on $\partial\Omega$, forces the last four integrals of (6.51) to vanish. Thus, we get

$$J_1 = -\frac{\pi}{\bar{k}}(S_{\gamma_1})_{12}(k) - \frac{\pi}{\bar{k}}(S_{\gamma_2})_{21}(k). \tag{6.54}$$

However, J_2 is not the same form as (3.49) and we won't get the nice equalities as in (3.54) and (3.55).

The approach we took here to try get a boundary integral equation that relates the off-diagonal entris of the scattering transform $S_\gamma(k)$ and Dirichlet-to-Neumann map is promising. We are hopeful that we can fix this problem in the near future.

6.4 The D-Bar Equatio, Computing γ and the Reconstruction Algorithm

We have successfully established a formula that computes the trace on $\partial\Omega$ of the exponentially growing solutions to the admittivity equation from the boundary data, this is step 1 of the reconstruction algorithm of γ from the Dirichlet-to-Neumann map. In this section, we give a $\overline{\partial}$ equation in the variable k for the exponentially growing solutions and construct admittivity γ using an appropriate equation that relates Q_γ to the exponentially growing solutions to the admittivity equation. This will take care of steps 3 and 4 of the reconstruction algorithm. Francini provides the necessary ingredients to do these two steps. From Theorem 13 of [Fra00], we have

$$\bar{\partial}_k M(z,k) = M(z,\bar{k})E_k(z)S_\gamma(k) \tag{6.55}$$

where $\bar{\partial}_k = \frac{1}{2}(\partial_{k_1} + i\partial_{k_2})$. So we have

$$M(z,k) = \bar{\partial}_k^{-1}(M(z,\bar{k})E_k(z)S_\gamma(k)),$$

where $\bar{\partial}_k^{-1} f(k)$ is the Cauchy transform given by

$$\bar{\partial}_k^{-1} f(k) = \frac{1}{\pi} \int_{\mathbf{R}^2} \frac{f(\xi)}{k - \xi} d\mu(\xi).$$

Francini [Fra00] established the following result for the recovery of Q_γ. For any $p > 0$,

$$Q_\gamma(z) = \lim_{k_0 \to \infty} \mu(B_p(0))^{-1} \int_E D_k M(z,k) d\mu(k),$$

where $E = \{k : |k - k_0| < p\}$.

So we can recover the potentials q and $\widetilde{q}$ from Q_γ

$$q(z) = \lim_{k_0 \to \infty} \mu(B_p(0))^{-1} \int_E \widetilde{q}(z) m_{11}(z,k) d\mu(k) \tag{6.56}$$

and

$$\widetilde{q}(z) = \lim_{k_0 \to \infty} \mu(B_p(0))^{-1} \int_E q(z) m_{22}(z,k) d\mu(k). \tag{6.57}$$

Chapter 7

FINAL REMARKS

This dissertation has two purposes: (1) develop properties of the off-diagonal entries of the scattering transform $S_\gamma(k)$, and (2) to develop propeties of the exponentially growing solutions to the admittivity equation. Moreover, we made contributions to the reconstruction algorithm of $\gamma \in W^{1,p}(\Omega)$.

We have seen in chapter 3 that the scattering transform and exponentially growing solutions to the conductivity equation are important objects of interest for the reconstruction algorithm of σ from the Dirichlet-to-Neumann map. Properties of both the scattering transform and the exponentially growing solutions to the conductivity equation are important from both the theoretical and applied points of view.

We saw that Francini's definition of the potential matrix did not lead to any symmetries of the off-diagonal entries of $S_\gamma(k)$. In chapter five we expand on the list of properties of the scattering transform $S_\gamma(k)$ using a modified definition of the potential matrix Q_γ. In particular, we saw that the off-diagonal entries of the scattering transform $S_\gamma(k)$ have the following properties under certain conditions

$$\overline{(S_\gamma)_{12}(k)} = (S_\gamma)_{21}(\bar{k}),$$

and

$$(S_\gamma)_{21}(e^{i\theta}k) = e^{-i\theta}(S_\gamma)_{21}(k) \quad \text{and} \quad (S_\gamma)_{12}(e^{i\theta}k) = e^{i\theta}(S_\gamma)_{12}(k).$$

In chapter six, we developed several properties involving the exponentially growing solutions to the admittivity equation. Some of these properties were useful in constructing a boundary integral equation for the exponentially growing solutions u to the admittivity equation.

We also did a promising investigation toward developing a boundary integral equation involving the off-diagonal entries of the scattering transform and the exponentially growing solutions ψ_{ij}. We are hopeful that in the near future we can use this idea to derive such a boundary integral equation.

In conclusion we have established the following steps in a reconstruction algorithm for the admittivity $\gamma \in W^{1,p}(\Omega)$:

- Equations for $(S_\gamma)_{12}(k)$ and $(S_\gamma)_{21}(k)$ in terms of ψ_{12} and ψ_{21} on $\partial\Omega$ were derived.
- A boundary integral equation for the exponentially growing solutions u in terms of the Dirichlet-to-Neumann data $\Lambda_\gamma - \Lambda_1$ was derived.
- A relation between u and ψ, where ψ is the exponentially growing Schrödinger solution was established.

To complete the reconstruction algorithm it still remains to establish a connection between $\psi_{12}|_{\partial\Omega}$, $\psi_{21}|_{\partial\Omega}$ and $u|_{\partial\Omega}$ or equations for ψ_{12} and ψ_{21} on $\partial\Omega$.

Appendix A

In this chapter we will review standard definitions of certain function spaces and their notations used in this thesis, they can be found in most graduate level such as [Eva98].

Let $\Omega \subseteq \mathbf{R}^2$ be a measurable set. As usual, let $L^p(\Omega)$ with $0 < p < \infty$ denotes the Lebesgue space, i.e, the space of equivalence classes of measurables functions f in Ω for which $|f|^p$ is integrable with the standard L^p norm

$$\|u\|_{L^p(\Omega)} = (\int_\Omega |u(x)|^p \, dx)^{1/p}.$$

Let $L^\infty(\Omega)$ denotes the space of essentially bounded measurable functions endowed with the norm

$$\|f\|_{L^\infty(\Omega)} = esssup \ f < \infty.$$

$L^p(\Omega)$ with $1 \leq p \leq \infty$, is a Branch space, i.e., a normed space that is complete with respect to the metric induced by its norm. $L^2(\Omega)$ is an example of Hilbert space is an inner product space which is complete under the norm induced by its inner product.

L^p_c denotes the subspace of L^p of measurable functions which are compactly supportly.

We will briefly discuss the most important function space of interest, the Sobolev space. Their importance lies in the fact that solutions of partial differential equations are naturally in Sobolev spaces rather than in the classical spaces of continuous functions and with the derivatives understood in the classical sense. Such function spaaces consist of elements whose weak derivatives of various orders live in various L^p spaces.

Let k be a nonnegative integer and $1 \leq p \leq \infty$. Define the Sobolev space, denoted by $W^{k,p}(\Omega)$, that consists of all locally summable functions $u : \Omega \to \mathbf{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Let $1 \leq p < \infty$. Define the norm of $u \in W^{k,p}(\Omega)$ to be

$$\|u\|_{W^{k,p}(\Omega)} = (\sum_{|\alpha \leq k|} \|D^\alpha u\|_{L^p(\Omega)})^{1/p}.$$

For $W^{k,\infty}(\Omega)$, we simply use the natural L^∞ norm inside the summation above.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space endowed with the above norm.

We write $H^k(\Omega) = W^{k,2}(\Omega)$, where k is a nonnegative integer. $H^k(\Omega)$ is a Hilbert space and $H^0(\Omega) = L^2(\Omega)$.

Let $s \geq 0$. We define the space

$$H^s(\Omega) = \left\{u \in L^2(\Omega) : (1 + |z|^2)^{s/2}\widehat{u}(z) \in L^2(\Omega)\right\}.$$

$H^s(\Omega)$ is a Hilbert space, endowed by the norm

$$\|u\|_{H^s(\Omega)} = (\int_\Omega (1 + |z|^2)^s \widehat{u}(z) dz)^{1/2.}$$

$C^\alpha(\Omega)$ is the usual Hölder space, i.e.,

$$C^\alpha(\Omega) = \left\{f : \|f\|_{l^\infty} + sup\frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty \ \forall \ x, y \in \Omega\right\}.$$

Let M_2 be the set of 2×2 matrices over $\mathbf{C}$. Denote the set $L^p(\mathbf{R}^2, M_2)$ the $L^p - space$ consisting of functions defined on $\mathbf{R}^2$ with M_2 consiting of elements in $L^p(\mathbf{R}^2)$.

Let $W^{1,p}_\beta(\Omega)$ be the weighted Sobolev space,

$$W^{1,p}_{-\beta}(\Omega) = \left\{(1 + |\cdot|)^{1/2} f \in W^{1,\bar{p}}(\Omega)\right\},$$

where $\beta > 1/\bar{p}$, $\frac{1}{p} + \frac{1}{\bar{p}} = \frac{1}{2}$.

Let m be a nonnegative integer. We write $C^m(\bar{\Omega})$ to denote the space of bounded dontinuous function on $\bar{\Omega}$ with continuous and bounded derivatives up to order m. We also write $C(\bar{\Omega}) = C^0(\bar{\Omega})$.

We will need the following standard inequalities.

Generalized Holder's Inequality: Leg $f_k \in L^{p_k}(\Omega)$ for $k = 1, 2, ...n$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n}$ where $p \geq 1$, then $\prod_{k=1}^n f_k \in L^p(\Omega)$. Moreover,

$$L^p(f_1 f_2 ... f_n) \leq L^{p_1}(f_1) L^{p_2}(f_2) ... L^{p_n}(f_n).$$

Minkowski's Inequality: If $f, g \in L^p(\Omega)$, then $f + g \in L^p(\Omega)$ and

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

A fundamental result in complex function theory is Liouville's Theorem which says that, the only bounded entire functions are the constant functions. Consequently, the only entire function in $L^p(\mathbf{R}^2)$, $1 \leq p < \infty$, is the zero function. We can use a generalized Liouville's Theorem to handle a class of functions in certain equations.

Define the differential operators ∂ and $\bar{\partial}$ by

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}), \quad \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}). \tag{A.1}$$

We say a function u is a pseudoanalytic in $\mathbf{C}$ with coefficients a, b if and only it satisfies the equation

$$\bar{\partial} u = a\bar{u} + bu, \ \ \mathbf{C} \tag{A.2}$$

Let $1 \leq p < \infty$. A key result is if $u \in L^p(\mathbf{R}^2)$ is a pseudoanalytic in $\mathbf{C}$ with the cofficients $a, b \in L^p(\mathbf{R}^2$, then $u = 0$ (see Theorem 3.1.3 of [Knu02]).

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