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A Three-Valued Interpretation for a Relevance Logic

In this paper an entailment relation which holds between certain propositions of the propositional calculus will be defined both syntactically and semantically. Some theorems about this relation will show why one could not follow Lewis to prove that a contradiction entails, for the notion of entailment discussed below, every proposition.

We will develop a system RC in which the primitive symbols are the five symbols

$$\vee \quad \cdot \quad - \quad (\quad)$$

and the propositional variables

$$p_1 \quad p_2 \quad p_3 \quad \ldots \quad .$$

The formation rules of RC are:

1) A variable standing alone is a well-formed formula (wff).

2) If A and B are well-formed (wf) then (A v B) is wf.

3) If A and B are wf then (A · B) is wf.

4) If A is wf then -A is wf.

5) If A is wf it is so in virtue of 1) - 4).

We will let capital letters with or without subscripts be variables which range over occurrences of wffs. Following Church (see Introduction to Mathematical Logic, pp. 135-6) we will say that X is the full disjunctive normal form of A (the FDNF of A), where $p_{i_1}, \ldots, p_{i_n}$ are all and the only propositional variable constituents of A, and $i_j < i_k$ if $j < k$, if and only if i) X is a disjunction of at most 2^n conjuncts such that each conjunction is identical to $(C_{m_1} \cdot (C_{m_2} \ldots C_{m_n}) \ldots)$ $(1 \leq m \leq 2^n)$, where C_{m_j} $(1 \leq j \leq n)$ is either p_{i_j} of $-p_{i_j}$ and ii) '((A · X) v (-A · -X))' is a tautology. By the full disjunctive normal form of A relative to B (the FDNF of A/B) we will mean the FDNF of A if every p_i in B is a p_i in A. If $p_{i_1}, \ldots, p_{i_n}$ are all and the only propositional variables which occur in B but not in A then by the FDNF of A/B we will mean the FDNF of

$(A \cdot ((p_{i_1} \vee -p_{i_1}) \cdot ((p_{i_2} \vee -p_{i_2}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots)$

Let us now define a relation which we will call syntactic-relevance-entailment and denote by ' → '. A → B if and only if

i) Every disjunct in the FDNF of A/B is a disjunct in the FDNF of B/A;

ii) There is at least one disjunct in the FDNF of A/B; and

iii) Every p_i in B is a p_i in A.

To define semantic-relevance-entailment, denoted by ' ⇒ ', we will use the notion of a valuation of a wff of RC. Let V be a valuation of A if V is a function which i) assigns 0, 1 or 2 to each p_i in A, ii) assigns the same value to different occurrences, if any, of the same p_i in A and iii) assigns 0, 1 or 2 to A as directed by the following tables:

v	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

·	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

	-
0	1
1	0
2	2

A ⇒ B if and only if

i) Every valuation that assigns 0 to A assigns 0 to B; and

ii) There is at least one valuation which assigns 0 to A.

The above notions of syntactic-relevance-entailment and semantic-relevance-entailment are extensionally equivalent. To show this (Theorem 1, below) we will make use of the following lemmas.

Lemma 1. If for every p_i in X $V(p_i) = 0$ or 1 then $V(X) = 0$ or 1. Proof: By strong induction on the number of symbols in X.

Lemma 2. If for every p_i in X $(V(p_i) = 0$ or 1, then $V(X) = 0$ if and only if V(the FDNF of X) = 0 and $V(X) = 1$ if and only if V(the FDNF of X) = 1. Proof: Standard result.

Lemma 3. If $V(X) = 0$ or 1 and p_i is a wf part of X then $V(p_i) = 0$ or 1. Proof: By using strong induction on the number of symbols in X we will prove that if p_i is a wf part of X and $V(p_i) = 2$ then $V(X) = 2$. i) Suppose that there is one symbol in X. Then $X = p_i$. If $V(p_i) = 2$ then $V(X) = 2$. ii) By the induction hypothesis if there are m symbols in X, where $m < n$, then if p_i is a wf part of X and $V(p_i) = 2$ then $V(X) = 2$. Consider a formula Y in which there are n symbols where $m < n$. We need to show that if there is a p_i in Y such that $V(p_i) = 2$ then $V(Y) = 2$. There are three cases to consider. a) $Y = (Y_1 \vee Y_2)$. Suppose there is a p_i in Y it must be either Y_1 or Y_2. If there is a p_i in Y_1 such that $V(p_i) = 2$ then by the induction hypothesis $V(Y_1) = 2$. But if $V(Y_1) = 2$ then $V(Y) = 2$. By similar reasoning if there is a p_i in Y_2 such that $V(p_i) = 2$ then $V(Y) = 2$. b) $Y = (Y_1 \cdot Y_2)$. Similar to case a). c) $Y = -Y_1$. Similar to case a).

Lemma 4. If V(the FDNF of A/B) = 0 then $V(A) = 0$. Proof: i) Suppose that every p_i in B occurs in A. Then the FDNF of A/B is identical to the FDNF of A. But if V(the FDNF of A) = 0 then by lemmas 2 and 3 $V(A) = 0$. ii) Suppose that $p_{i_1}, \ldots, p_{i_n}$ are all and the only variables that occur in B but not in A. Then the FDNF of A/B is identical to the FDNF of $(A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots)$. By lemmas 2 and 3 if V(the FDNF of A/B) = 0 then $V((A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots) = 0$ and thus (by the tables) $V(A) = 0$.

Lemma 5. If V(the FDNF of A/B) = 1 then V(A) = 1. Proof: Similar to the proof for Lemma 4. (Note that if $V((A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots) = 1$ then V(A) = 1.)

Lemma 6. If the FDNF of A/B is non-empty then there is a valuation which assigns 0 to A. Proof: If the FDNF of A/B is non-empty let $(C_{1_1} \cdot (C_{1_2} \cdot \ldots \cdot C_{1_n}) \ldots)$ be the left-most disjunct of the FDNF of A/B, where $p_{i_1}, \ldots, p_{i_n}$ are the variables which occur in A or B. If $C_{1_r} = p_{i_r}$ let $V(p_{i_r}) = 0$; if $C_{1_r} = -p_{i_r}$ let $V(p_{i_r}) = 1$. Then $V((C_{1_1} \cdot (C_{1_2} \cdot \ldots \cdot C_{1_n}) \ldots) = 0$. Since $C_{j_r} = p_{i_r}$ or $-p_{i_r}$ $(1 \leq j \leq 2^n;\ 1 \leq r \leq n)$ V(any disjunct in the FDNF of A/B) = 0 or 1. So V(the FDNF of A/B) = 0. By Lemma 4 V(A) = 0.

Lemma 7. If there is a valuation V such that V(A) = 0 then the FDNF of A/B is non-empty. Proof: Suppose V(A) = 0. Then by lemmas 2 and 3 V(the FDNF of A) = 0. Since there is nothing assigned to the empty symbol by V, the FDNF of A is non-empty. If the FDNF of A is non-empty then the FDNF of $(A \cdot ((p_{i_1} \vee -p_{i_1}) \ldots (p_{i_n} \vee -p_{i_n}) \ldots)$ is non-empty, where p_{i_j} $(1 \leq j \leq n)$ does not occur in A. So the FDNF of A/B is non-empty if V(A) = 0.

Theorem 1. $A \rightarrow B$ if and only if $A \Rrightarrow B$. Proof: Consider the three conditions: a) every disjunct in the FDNF of A/B is a disjunct in the FDNF of B/A,

b) there is at least one disjunct in the FDNF of A/B and c) every p_i in B is a p_i in A. We must show that these conditions are met if and only if the following two conditions are met: d) every valuation which assigns 0 to A assigns 0 to B and e) there is a valuation which assigns 0 to A.

i). (If a), b) and c) then d).) Assume that a), b) and c) are true and that V(A) = 0. By lemmas 2 and 3 V(the FDNF of A) = 0. Since c) is true the FDNF of A is identical to the FDNF of A/B. So V(the FDNF of A/B) = 0. So V assigns 0 to at least one of the disjuncts of the FDNF of A/B. By a) V assigns 0 to at least one of the disjuncts of the FDNF of B/A. But then V assigns 0 or 1 to each of the disjuncts of the FDNF of B/A and thus V assigns 0 to the FDNF of B/A. By Lemma 4 if V(the FDNF of B/A) = 0 then V(B) = 0. ii) (If a), b) and c) then e).) Follows from Lemma 6.

iii) (If d) and e) then a).) We will show that if a) is false then d) is false. Suppose there is a disjunct of the FDNF of A/B which is not a disjunct of the FDNF of B/A. Let V assign 0 to this disjunct. Then V assigns 1 to every other disjunct in the FDNF of A/B and 1 to every disjunct in the FDNF of B/A. So V(the FDNF of A/B) = 0 and V(the FDNF of B/A) = 1. By lemmas 4 and 5 V(A) = 0 and V(B) $\neq$ 0. iv) (If d) and e) then b).) Follows from Lemma 7. v) (If d) and e) then c).) We will show that if c) is false then d) is false. Assume that p_i is in B but not in A. Let V(A) = 0 and $V(p_1)$ = 2. By Lemma 3 V(B) = 2.

Theorem 2. (Simplification) If there is a valuation which assigns 0 to $(A \cdot B)$ then $(A \cdot B) \Rrightarrow A$. Proof: By examination of tables.

Theorem 3. (Commutation) If there is a valuation which assigns 0 to $(A \cdot B)$ then $(A \cdot B) \Rrightarrow (B \cdot A)$. Proof: By examination of tables.

Theorem 4. (Addition) If there is a valuation which assigns 0 to A and if every p_i in B is a p_i in A then $A \Rightarrow A \vee B$. Proof: Assume that $V(A) = 0$ and that every p_i in B is a p_i in A. By lemmas 1 and 3 $V(B) = 0$ or 1. So $V(A \vee B) = 0$.

Theorem 5. (Adjunction) If $A \Rightarrow B$ and $A \Rightarrow C$ then $A \Rightarrow (B \cdot C)$. Proof: By examination of tables.

Theorem 6. (Disjunctive Syllogism) If there is a valuation which assigns 0 to $(-A \cdot (A \vee B))$ then $(-A \cdot (A \vee B)) \Rightarrow B$. Proof: By examination of tables.

Theorem 7. (Transitivity of Entailment) If $A \Rightarrow B$ and $B \Rightarrow C$ then $A \Rightarrow C$. Proof: i) if $A \Rightarrow B$ then there is a valuation which assigns 0 to A. ii) If $V(A) = 0$ then if $A \Rightarrow B$ $V(B) = 0$. But if $V(B) = 0$ and $B \Rightarrow C$ then $V(C) = 0$. So if $V(A) = 0$ and the antecedent of Theorem 7 is true $V(C) = 0$.