A DISCHARGE FORMULA FOR FLOW IN STRAIGHT ALLUVIAL CHANNELS

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I. INTRODUCTION

The original title of this research was "Analytical Study of Alluvial Channel Roughness", which was a research project granted to the first author by the National Science Foundation. The purpose of this research was to find a suitable formula to determine more accurately the mean velocity of flow, and thereby the discharge of flow, in alluvial channels. At the beginning of the research, the authors intended to study the variation of Manning's roughness coefficient or of Chezy's discharge coefficient as a function of the characteristics of the flow and properties of the sediment. It was found later that such an approach is not yet feasible. A new velocity formula was attempted (1), the result of which is presented in this paper.

In order to understand the problem of determining the mean velocity of an alluvial stream more clearly, it is necessary to study

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the velocity distribution for turbulent flow in pipes, in open channels and even in the turbulent boundary layer, and also to study the mechanics of sediment transport. After considerable review of literature the writers came to the conclusion that a theoretical approach to the problem cannot be obtained at the present time, therefore an empirical approach was adopted. In order to facilitate the empirical correlation the method of dimensional analysis was used so that none of the significant parameters would be omitted. A consistent empirical correlation was found and this has been further reduced to an exponential formula for practical application.

II. LITERATURE REVIEW

Most literature on mean velocity equations, therefore on discharge equations is for clear flow in rigid conduits, either pipes or open channels. About 1768 Chezy (2) proposed a method of estimating the mean velocity of a stream by comparing the flow conditions with those of another having similar characteristics. Such a proposition has been customarily written in a form known as Chezy's formula,

$$V = C\sqrt{RS} \tag{1}$$

in which V is the mean velocity, C the Chezy discharge coefficient, R the hydraulic radius, and S the slope of the channel.

In 1869 Ganguillet and Kutter suggested a formula for determining Chezy's C:

$$C = \frac{a + \frac{b}{n} + \frac{m}{s}}{m + \left(a + \frac{m}{s}\right)\frac{n}{\sqrt{R}}} \tag{2}$$

in which a, b, and m are constants and n is a roughness factor.

In 1889, Manning (3) proposed several formulas for estimating the mean velocity of turbulent flow in conduits. The following well known Manning's formula was included in his original paper. However, Manning did not recommend its use because the equation

$$V = MR^{2/3}S^{1/2} \tag{3}$$

is not homogeneous.

It is currently written for the English system as

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \qquad (4)$$

in which M is an empirical constant depending upon the boundary roughness of the conduit and n is the roughness factor.

The writers introduce these commonly-used formulas here to demonstrate that the exponents of the hydraulic radius and the slope are not the same in these equations. Additional information regarding empirical velocity formulas can be found in the book "Hidraulik" written by Dr. S. Kolupaila (4). Kolupaila shows that numerous exponents of the hydraulic radius and of the slope have been proposed in the past.

From an analytical point of view the mean velocity of a turbulent flow depends upon the velocity distribution, which is related to the mechanics of turbulent flow. The equation of motion for turbulent flow is known as the Reynolds equation which differs from the Navier-Stokes equation by additional terms called the Reynolds stresses. The solutions of the Reynolds equations will represent properly the turbulent flow. Since the Reynolds equations are not sufficient to determine the Reynolds stresses, additional equations have to be obtained either through hypothesis or through experimental measurements.

Among the various formulas of velocity distribution proposed for turbulent flow, the logarithmic law is frequently used by hydraulic engineers. A brief review of this law may be helpful to understand its limitation of application. The logarithmic law can be obtained either from Prandtl's hypotheses (5) of mixing length by assuming that, near the wall, the mixing length is linearly proportional to the distance from the wall and the shear stress is constant, or from Karman's similarity hypothesis (6) by assuming that the mixing length is only a function of the velocity distribution and the shear stress is constant. Therefore, the logarithmic law is for turbulent flow near rigid boundaries. It can be written as (7)

$$\frac{u}{V_*} = \frac{1}{K} \ln \frac{y}{y_o} \tag{5}$$

in which u is the local mean velocity along the flow direction at a distance y from the boundary, V_* is the shear velocity $\sqrt{\tau_o/\rho}$ in which τ_o is the local boundary shear, K is the so-called universal constant and the value of y_o is dependent upon the hydraulic roughness of the boundary.

From Nikuradse's data for turbulent flow in pipes, it can be found that in case that $V_* K_s/\nu$ is less than about 3.5, in which K_s is the size of the sand used in the test, the boundary can be classified as hydraulically smooth and Eq 5 can be written for flow outside the laminar sub-layer as (7)

$$\frac{u}{V_*} = \frac{2.3}{K} \log_{10} \frac{V_* y}{\nu} + 5.5 \tag{6}$$

In case that $V_* K_s/\nu$ is greater than about 70, the boundary can be classified as hydraulically rough, and Eq 5 can be written as (7).

$$\frac{u}{V_*} = \frac{2.3}{K} \log \frac{y}{K_s} + 8.5 \tag{7}$$

Nikuradso (8) found that the logarithmic law is not applicable to the flow near the center of the pipe ; which is self evident according to the assumptions used in the derivation of the law. If the logarithmic law were exact to describe the velocity distribution of turbulent flow in pipes, the total discharge, and hence the mean velocity of the flow, could be determined by integration through the use of the logarithmic law. It was found that the constants in the resultant equations have to be modified in order to yield satisfactory results. In general the formula of mean velocity for turbulent flow in a smooth pipe is (9)

$$\frac{V}{V_*} = C_1 \log_{10} \frac{V_* R}{\nu} + C_2 \tag{8}$$

and that for turbulent flow in a rough pipe is (9)

$$\frac{V}{V_*} = C_1 \log_{10} \frac{R}{K_s} + C_3 \tag{9}$$

in which C_1 , C_2 and C_3 are constants and R is the hydraulic radius.

Keulegan (9) applied Nikuradse's results to open channel flow, he showed that when the hydraulic radius is used as the characteristic length, the Nikuradse formula for pipe flow can be applied to open channel flow. However, Powell (10) found that because of the existance of a free surface in the open channel flow such an extension of Nikuradse's work to open channels cannot be done successfully. Additional information for flow in open channels composed of artificial roughness element on the boundary can be found from the works of Albertson and Robinson (11), Sayre (12) and Johnson (13).

In the foregoing review of the logarithmic law, there are two important points to the present study; (1) Although the logarithmic law for turbulent flow near rigid boundaries has been verified by experimentation, the Karman-Prandtl hypotheses have not been proved to be theoretically sound, (2) the classification of the boundary roughness is in accordance with the concept of the boundary layer.

Millikan (14) raised some doubts about the Karman-Prandtl hypotheses and showed that without employing these hypotheses the velocity distribution of turbulent flow in pipes or channels follows the logarithmic law in the overlap zone where the "law of wall" and the "velocity-defect law" are both applicable. The law of wall, which is due to Prandtl by use of dimensional analysis, can be written as (14):

$$\frac{u}{V_*} = F_1\left(\frac{V_* y}{\nu}, \frac{y}{K_s}\right) \quad (y \rightarrow 0) \tag{10}$$

The "velocity-defect law" is essentially empirical, first enunicated in its general form by Karman and can be written as (14)

$$\frac{U_{max} - u}{V_*} = G\left(\frac{y}{h}\right) \quad \left(\frac{y}{h} \not\rightarrow 0\right) \tag{11}$$

in which $u = U_{max}$ at $y = h$, and h is the value of y at the center of the channel or pipe.

Further discussion on velocity distribution will be presented later in connection with the review of the turbulent boundary layer. The following remarks may be related to the classification of boundary roughness:

In the case of a rough boundary, the effect of viscosity on the velocity distribution can be neglected. Any discharge formula, such as Chezy's or Manning's which does not consider the effect of viscosity on the mean velocity, hence on the discharge, is only applicable to the case of turbulent flow along rough boundaries.

In the case of a smooth boundary, the effect of the roughness elements on the velocity distribution can be neglected. In addition to Eqs 6 and 8, which resulted from the Karman-Prandtl hypothesis, there is another formula known as the $\frac{1}{7}$ - power velocity-distribution law (15):

$$\frac{u}{V_*} = 8.74 \left(\frac{V_* y}{\nu}\right)^{1/7} \tag{12}$$

Equation 12 was first discovered by Prandtl from the following Blasuis' empirical law of friction (16):

$$f = \frac{0.316}{\left(\frac{VD}{\nu}\right)^{1/4}} \tag{13}$$

in which

$$f = 8\left(\frac{V_*}{V}\right)^2 \tag{14}$$

and D is the diameter of the pipe.

According to Schlichting (15), the exponents in Eqs 12 and 13 are not constants, but dependent upon the Reynolds number of the mean flow. Equation 13 can be changed into an exponential type of discharge formula which will be discussed later.

Since the problem of the mean velocity and the velocity distribution of turbulent flow in open channels is essentially one of a turbulent boundary layer, a brief review of the literature on the turbulent boundary layer along a flat plate at constant pressure may shed some light on the problem of velocity distribution in open channels.

From extensive wind tunnel measurements it is found that the mixing length theory has many limitations and inconsistencies. At the present time, scientists seem to be in favor of statistical mechanics to study turbulence. A completely satisfactory theory of turbulence is not available; scientists are seeking for laboratory data so that some theory of turbulence can be formed.

It has been found (17) in the wind tunnel that the logarithmic law is valid only within about 15 per cent of the thickness of the turbulent boundary layer. According to Clauser (17), the flow within the turbulent boundary layer can be divided into two regions. In the inner region, the law of wall is applicable; in the outer region, the velocity defect law is applicable. In the overlapping zone where both the law of wall and the velocity-defect law are applicable, the logarithmic velocity distribution prevails, which is similar to Millikan's conclusion for turbulent flow in pipes and in channels.

Also according to Clauser the inner portion of the layer responds to the wall shear much faster than the outer portion. While the inner portion completes its response within a few boundary layer thicknesses traveled, the outer portion takes tens or even hundreds of the boundary layer thickness for a corresponding response. A comparison of the response distance and mode of response to distrubances of various kinds and intensities confirms that a boundary layer is a truly non-linear phenomenon. Consequently, progress cannot be made by applying a linear concept of predeterminable response distances or times. Since the outer portion does not respond to the wall shear very quickly, the velocity distribution in the outer portion depends also on the history of the flow. Although the law of wall has been found to be independent of the pressure gradient along the boundary, it has not been proven to be applicable to the case where the boundary is movable or flexible, such as the case of alluvial boundaries.

In brief summary it can be stated that, for the time being, there is no satisfactory theory of turbulent flow available so that the velocity distribution in turbulent flow can be calculated or predicted. Furthermore, since the flow is non-linear in nature, it is very doubtful that a theoretical and exact solution of the turbulent flow problem will ever be available, even though an approximate solution may be possible after extensive experimentation.

Another factor which is important in the study of the mean velocity in alluvial channels, but is absent from flow in rigid channels, is the sediment transport. For flow transporting sediment, there are two major problems involved: (1) the amount of sediment transport, (2) the problem of channel roughness and its effect on the discharge of the flow. There are several formulas (18), (19), (20) for estimating the amount of sediment transported. On the other hand, there is very little literature proposing velocity formulas for alluvial streams. In case the bottom is plane, the alluvial boundary has been treated as a rigid one. For example, Strickler (21) proposed that Manning's roughness factor can be expressed as a function of the sediment size for small boulders and cobbles:

$$n = 0.0160\, d^{1/6} \qquad d \text{ in mm} \tag{15a}$$

or

$$n = 0.039\, d^{1/6} \qquad d \text{ in ft} \tag{15b}$$

One of the major difficulties in determining the mean velocity of alluvial streams is that the bed configuration changes with the flow condition. Consequently the bed roughness, which affects the velocity, changes with the flow condition. In 1950, Einstein and Barbarossa (22) proposed that the boundary shear of a dune bed be divided into two portions: (a) that pertaining to the grain roughness and, (b) that pertaining to the dune roughness. Although such an approach seems logical, its application to practical problems is still very limited.

Vanoni and Brooks (23) have shown that suspended load can cause a reduction in the resistance coefficient. They claim that the discharge and sediment load cannot be expressed as unique functions of the depth, slope and sand size. However, this view is not shared by other investigators.

In conclusion it can be stated that the theory of turbulent flow is still not completed even for flow near rigid boundaries. Its development for the case of a flow near a movable boundary seems even more remote. Moreover, the effect of sediment transport on the resistance coefficient is still unknown. Hence no theoretical analysis can be made at the present of the problem of mean velocity of alluvial streams. Therefore, an empirical correlation seems to be desirable for engineering purposes.

III. ANALYSIS OF THE PROBLEM OF MEAN VELOCITY IN STRAIGHT ALLUVIAL CHANNELS

In the case of flow carrying sediment, the change of flow causes not only the change of sediment transport but also the change of bed configuration. The phenomenon of sediment transported can be described by assuming that: (1) the bed material is granular and cohesionless (2) the amount of supply of the sediment is equal to the amount of sediment transport, and (3) the flow is turbulent, steady and uniform. Let the bed be initially plane at a small discharge with no sediment moving. As the discharge increases, the first movement of sand grains will be rolling and sliding -- occurring intermittently in spots. As the discharge is further increased, the movement of sediment becomes more intense. It can be stated that statistically there is a critical condition under which the movement of sediment begins. If the discharge is increased still further ripples appear on the bed at a certain stage. A ripple bed is characterized by a rather regular wave pattern. The amplitudes of the ripples are usually small compared to their wave lengths. The characterisitics of ripples are such that they eventually will become asymmetric, as demonstrated by Exner (24). At a later stage, sand dunes appear on the bed. A dune bed is usually characterized by a long sloping upstream face with an abrupt downstream face. The sediment is eroded along the upstream face and deposited in the trough. The pattern of sand dunes is not as regular as that of ripples. The change of bed surface from plane to ripples and dunes causes an abrupt change of bed roughness.

On further increase of the discharge, at a certain stage the bed becomes flat. The bed roughness at this stage is much less than that of dunes and is about the same as that of a plane bed. This is a transition stage between dunes and antidunes.

The case of antidune is such that the sediment is deposited on the upstream face of the sand wave ~~dune~~ and eroded from the downstream face of the sand wave ~~dune~~, consequently the sand wave ~~dune~~ moves upstream while the sediment is transported downstream.

The various bed configurations can be estimated from Fig. 1 (25) for a given flow depth, slope, bed material size, and fluid temperature. In Fig. 1, Shield's (26) criterion for the beginning of motion and Liu's (27) criterion for the beginning of ripples respectively are shown. A modified criterion for beginning of ripples and criteria for the formation of dunes and for transition proposed by Albertson, Simons and Richardson (28) are also shown. For lack of an exact and unified definition of various bed configurations, data from various sources do not always agree with the classification shown in Fig. 1. However, Fig. 1 is adopted in this study for the reason of lack of a better classification.

The change of resistance coefficient f as a result of the change of bed configuration is illustrated by Fig. 2. The curves are for pipe flow after Nikuradse, and the points are data pertaining to the movable bed taken from Report No. 17 by the U. S. Waterways Experimental Station (29). As long as the bed is plane, the variation of the resistance coefficient as the flow changes is similar to that of pipe flow. The sudden increase of the resistance coefficient indicated by the points occurs as sand waves appear on the bed. From this it is easy to see that application of the Nikuradse approach to the case of alluvial channels is not likely to be successful.

The exponential type of discharge formula is already in existance for flow in rigid channels. For example, the Blasius equation which is for turbulent flow along a smooth boundary can be written in exponential form as

$$V = \frac{56}{{}^{1/7}} R^{5/7} S^{4/7} \tag{16}$$

which for $t = 65^0F$. $V(water) = 1.12 \times 10^{-5}\ ft^2/sec$.

Therefore $$V = 285\ R^{0.714}\ S^{0.57} \tag{17}$$

Note that according to Eq 16, a variation of temperature of 20^0F from 65^0F changes the mean velocity about 4 per cent. Hence the effect of water temperature on mean velocity is normally negligible. The Manning formula which is for turbulent flow near a rough boundary is also an exponential formula. Note that these two formulas are for extreme cases, and the exponents are not the same. It is possible that the exponents of the discharge formula for turbulent flow in the transition region have other values. In general the exponential type of discharge formula can be written as

$$V = C_r R^x S^y \tag{18}$$

in which C_r is an empirical coefficient and x and y are pure numbers. In Blasius' formula $x = 5/7$ and $y = 4/7$, and in Manning's formula $x = 2/3$ and $y = 1/2$.

For two-dimensional, steady, uniform flow the depth of flow can be considered dependent mainly upon the following variables: q is the unit discharge of the flow, S is the slope of the channel, ρ the fluid density, μ the fluid viscosity, g the gravitational constant, d the mean size of the bed material, σ the standard deviation of the bed material, $\Delta\gamma_s$ the difference in specific weight between the bed material and the fluid, and η the shape factor of the sediment, namely

$$D = \phi_1(q, S, \rho, \mu, g, d, \sigma, \Delta\gamma_s, \eta) \quad (19)$$

Since q = DV , and w = fall velocity = w(d , ρ , μ , $\Delta\gamma_s$, η)

Eq 19 can be written as

$$D = \phi_2(S, V, \rho, \mu, g, d, \sigma, \Delta\gamma_s, w) \quad (20)$$

By use of π -theorem with D , V , and ρ as repeating variables, Eq 20 can be written as

$$\phi_3\left(S, \frac{V}{\sqrt{gD}}, \frac{VD}{\nu}, \frac{d}{D}, \frac{\sigma}{D}, \frac{V^2\rho}{\Delta\gamma_s D}, \frac{V}{w}\right) = 0 \quad (21)$$

If the effect of sediment mixture on the flow depth is considered to be secondary, the term σ/D can be omitted from the equation.

Notice that

$$\frac{V}{\sqrt{gD}} = \frac{V}{V_*}\sqrt{S} \quad (22a)$$

$$\frac{VD}{\nu} = \frac{V}{V_*} \frac{V_* d}{\nu} \frac{D}{d} \qquad (22b)$$

$$\frac{\rho V^2}{\Delta\gamma_s D} = \frac{\rho V_*^2}{\Delta\gamma_s d} \left(\frac{V}{V_*}\right)^2 \frac{d}{D} \qquad (22c)$$

$$\frac{V}{w} = \frac{V}{V_*} \frac{\frac{V_* d}{\nu}}{\frac{wd}{\nu}} \qquad (22d)$$

therefore a new set of dimensionless terms can be substituted into Eq 21 such as

$$\phi_4\left(S, \frac{V}{V_*}, \frac{V_* d}{\nu}, \frac{d}{R_b}, \frac{\tau_b}{\Delta\gamma_s d}, \frac{wd}{\nu}\right) = 0 \qquad (23)$$

In which $\tau_b = \rho V_*^2$, and the hydraulic radius pertaining to the bed R_b is substituted for D. In case the sediment is spherical wd/ν can be omitted from Eq 23 (25). In this discussion, Eq 23 is used as a guide in the correlation of data.

IV. EMPIRICAL CORRELATION OF DATA

No specific laboratory work was done by the authors for this research. As much as possible existing data were collected. Both laboratory data and canal data were used. Although most of the data were for flow having bed load only, considerable data for flow having both suspended load and bed load were used. The depth of flow ranged from a few inches to several feet. The velocity of flow varied from less than one foot per second to six feet per second. The slope of flow varied from 0.0004 to 0.028. The size of sediment varied from .01 mm to 3 or 4 inches. Both uniform bed material and graded bed material were used. The variation of viscosity and sediment density of these data were not appreciable. The effect of side wall was corrected according to the standard procedure (30).

As explained earlier a theoretical treatment of the problem of mean velocity is not possible at the present time. The result of this study, therefore has been obtained from empirical correlations based upon physical reasoning, dimensional analysis, and the mechanics of the boundary layer. The drawback of using empirical correlation is that usually the parameters cannot be explained either theoretically or physically.

The authors used three parameters $V_* d/\nu$,

$$\frac{\frac{V}{V_*} \frac{\tau_b}{\Delta\gamma_s d} S^{\lambda}}{\left(\frac{R_b}{d}\right)^m F_r^{\,N}}$$

and wd/ν to correlate these data.

The first parameter is the shear-velocity Reynolds number and the second dimensionless parameter, which can be abbreviated as K, needs some explanation. Note that V/V_* , S and F_r are related to one another. These dimensionless parameters were evolved from a plot made by the first author (31) in his previous study of the roughness of alluvial beds. In this earlier study only $\frac{V_* d}{\nu}$, $\frac{V}{V_*} \frac{Tb}{\Delta \gamma_s d}$, $\frac{w d}{\nu}$ were used. The term $\frac{V}{V_*} \frac{Tb}{\Delta \gamma_s d}$ was interpreted as the tractive force divided by the submerged weight of the particle multiplied by a resistance coefficient V_*/V . In addition to these parameters R_b , S and F_r were added for the following reasons:

1. To conform with the existing knowledge of boundary resistance,

2. To correlate the data consistantly.

For any given constant value of wd/ν , the data, when plotted according to $V_* d/\nu$ against K shown in Fig. 3, were found to fall on two straight lines depending upon whether the bed is a plane bed or a dune bed. The condition at which the plane bed changes into wavy bed can be estimated according to Fig. 1 (ripples are considered as incipient dunes). The position of these straight lines depend also upon the third variable wd/ν . For clarity, straight lines pertaining to the plane bed have been plotted separately from those pertaining to the dune bed. These straight lines for a plane bed are shown in Fig. 4. Their slope is 1:0.555, and the positions are shown in Fig. 5 according to the third variable wd/ν , which varies from 0.0012 to 104,000. A general equation can be written for the data for the plane bed

$$\frac{V_* d}{\nu} = A \left[\frac{\frac{V}{V_*} \frac{\tau_b}{\Delta \gamma_s d} S^{\lambda}}{\left(\frac{R_b}{d}\right)^m F_r^N} \right]^{\Omega} \tag{24}$$

in which Ω is equal to 0.555 , A is a function of the third variable $\frac{wd}{\nu}$ as shown in Fig. 5, and λ , m and N are pure numbers shown in Fig. 6 which were obtained empirically as functions of the mean size of the bed material.

From Fig. 5, for $wd/\nu > 1000$, the factor A can be expressed as

$$A = \epsilon \frac{wd}{\nu} \tag{25}$$

in which ϵ has an approximate value of 0.39, and is dependent upon the shape factor and also the fall-velocity Reynolds number of the grain of the bed material, *and*

$$w = \sqrt{\frac{4}{3} \frac{1}{C_D} \frac{\rho_s - \rho}{\rho} g d}$$

in which C_D is the drag coefficient of the grain of the bed material.

When Eq 25 is substituted in Eq 24, the result is:

$$V = 3.35 \left(\frac{C_D}{\epsilon^2}\right)^{\frac{4.5}{2}} d^{\frac{2\Omega(1-m)-1}{2\Omega(1-N)}} R_b^{\frac{1-\Omega(1+N-2m)}{2\Omega(1-N)}} S^{\frac{1-\Omega(1+2\lambda)}{2\Omega(1-N)}} \tag{26}$$

Eq 26 does not contain the factor of viscosity, μ or ν , which is reasonable for turbulent flow near rough boundaries. If the values of λ , m , N , and Ω for plane bed of very large bed material, that is, $\Omega = 0.555$, $m = \frac{1}{6}$, $N = 0.6$ and $\lambda = 0.2$ are substitued in Eq 26, the result will be

$$V = 3.35\left(\frac{C_D}{\epsilon^2}\right)^{\frac{4.5}{2}} d^{-\frac{1}{6}} R_b^{2/3} S^{1/2} \tag{27}$$

The exponents of d, R_b and S are the same as those given by the Manning-Strickler formula. If the drag coefficient C_D is chosen as 0.49, together with $\epsilon = 0.39$, Eq 27 will become the ordinary Manning-Strickler formula. Theoretically both the coefficient ϵ and the drag coefficient C_D are dependent upon the shape and the fall-velocity Reynolds number of the grain of the bed material. The fact that both ϵ and C_D are of the same order of magnitude and the fact that the exponent of ϵ is twice as large as that of C_D may explain why the mean velocity may be proportional to $C_D^{2.25}$ The factors ϵ and C_D in Eq 26 can be used to explain why the Strickler's coefficient is different from that of Chang's (32), which is

$$n = 0.0166 d^{1/6} \qquad d \text{ in mm} \tag{28}$$

From Fig. 5, for $\frac{wd}{\nu} < 1$, the factor A can be written as

$$A = \Theta\left(\frac{wd}{\nu}\right)^p \tag{29}$$

in which Θ is a constant, approximately of 3.4, p is the slope of the curve, approximately to be $\frac{1}{2}$ and $w = \frac{1}{18}\,\frac{\rho_s - \rho}{\rho}\,gd^2$ for spherical grains.

Substituting Eq 29 in Eq 24 with $\Theta = 3.4$ yields

$$V = 56\, \nu^{\frac{2p-1}{\Omega(1-N)}}\, d^{\frac{1+\Omega(1-m)-3p}{\Omega(1-N)}}\, R_b^{\frac{1-\Omega(1+N-2m)}{2\Omega(1-N)}}\, S^{\frac{1-\Omega(1+2\lambda)}{2\Omega(1-N)}} \qquad (30)$$

For the limiting values of Ω, N, m and λ for plane bed composed of very small bed material, that is, $\Omega = 0.555$, $N = 0.8$, $m = \frac{1}{7}$ and $\lambda = 0.287$, as used, the exponent of R_b reduces to 0.72, the exponent of S to 0.57, the exponent of d reduces to zero at $p = 0.492$, and consequently the exponent of ν reduces to $\frac{1}{7}$. Equation 30 is then reduced to the Blasuis equation (Eq 16) for turbulent flow near smooth boundaries. Should the value of p be $\frac{1}{2}$, the exponent of ν is then zero, which agrees with previous discussion that the effect of viscosity on the mean velocity is very small and can be neglected.

A similar correlation can be found for dune bed as shown in Figs. 7 and 8. The data shown on Figs. 7 and 8 can be represented also by Eq 24, except that Ω for dune bed is 0.565, and the exponents λ, m and N for dune bed are as shown also in Fig. 6.

Equation 24 is considered to be the general equation representing the flow characteristics of alluvial streams. Although Eq 24 is dimensionally homogeneous, it is not convenient to use. A further simplification of Eq 24 is discussed in the next chapter.

V. PROPOSED DISCHARGE FORMULA FOR ALLUVIAL STREAMS

For simplification Eq 24 can be reduced to

$$V = C_a R_b^x S^y \tag{31}$$

in which C_a is the discharge coefficient for alluvial streams and can be computed by Eq 32

$$C_a = \bar{\Psi} \, d^{\frac{1+\Omega(1-m)}{\Omega(1-N)}} \tag{32a}$$

in which

$$\bar{\Psi} = \left[\frac{P}{\sqrt[\Omega]{A}} \right]^{\frac{1}{1-N}} \tag{32b}$$

and

$$P = \frac{g^{\frac{1+\Omega(1-N)}{2\Omega}}}{\sqrt[\Omega]{\nu}} \quad \frac{\rho_s - \rho}{\rho} \tag{32c}$$

and

$$A = f\left(\frac{wd}{\nu}\right) \tag{32d}$$

It can be seen from Eq 32 that the discharge coefficient C_a is a function of d, ρ_s, ρ, g, and ν. Furthermore, the bed material is generally composed of a mixture of non-spherical grains and the coefficient C_a depends also upon the shape factor and the standard deviation of the bed material. In Eq 31 x and y are pure numbers and can be computed from Eqs 33 and 34 respectively,

$$x = \frac{1 - \Omega(1 + N - 2m)}{2\Omega(1 - N)} \tag{33}$$

and

$$y = \frac{1 - \Omega(1 + 2\lambda)}{2\Omega(1 - N)} \tag{34}$$

in which λ, m and N are shown in Fig. 6 and

Ω = 0.555 for plane bed

Ω = 0.565 for dune bed

and when the values of Ω, λ, m and N are substituted according to the bed configuration (plane bed or dune bed), the results of the exponents x and y are shown in Figs. 9 and 10 respectively as functions of the bed configuration and the size of the bed material. Since x and y are pure numbers, it is reasonable to assume that they depend upon some dimensionless parameter, rather than upon the size of the bed material alone. The dimensionless parameter which is still unknown should be directly related to the boundary conditions, and/or to the flow conditions. The unknown dimensionless parameter may be some combination of those given in Eq 23, with the possible exception of V/V_*. Such a dimensionless parameter has not been attempted by the authors. On the other hand, since the choice of x and y depends partly upon bed configuration which is governed by two dimensionless parameters, V_*/w and wd/ν, the effect of the hydraulic boundary condition on the choice of x and y has been partly, if not entirely, considered. Further research is needed to express the exponents x and y as functions of certain dimensionless parameters.

Note that in Fig. 9, the variation of the exponent x against the bed material size d for dune bed is opposite to that for plane bed. Both the exponent x for plane bed and that for dune bed are $\frac{2}{3}$ when the bed material size d is greater than 4 mm (the exponent x of $\frac{2}{3}$ is the same as appeared in Manning's formula). For plane bed, x increases as d decreases for the bed material size smaller than 4 mm. The x-value reaches an upper limit of $\frac{5}{7}$ as d becomes less than 0.2 mm (the exponent x of $\frac{5}{7}$ is the same as appeared in the Blasius formula). For dune bed, x decreases as d decreases. The value of x is 0.35 when the value of d is 0.01 mm.

Note that in Fig. 10, the variation of the exponent y against the sediment size d for dune bed is also opposite to that for plane bed. The exponent y for plane bed is $\frac{1}{2}$ when the bed material size is greater than 4 mm (the y-value of $\frac{1}{2}$ is the same as appeared in Manning's formula). As the size of the bed material decreases, the exponent y for plane bed increases for d smaller than 4 mm. The y-value reaches an upper limit of 0.57 when d becomes 0.1 mm or smaller (the y-value of 0.57 is the same as appeared in the Blasius formula). The exponent y for dune bed becomes $\frac{1}{2}$ when the bed material size is 20 mm or greater. For dune bed, as the size of the bed material decreases the exponent y decreases. The y-value is 0.30 when the d-value is 0.1 mm or smaller.

That the exponent y for dune bed is normally below $\frac{1}{2}$ may need some discussion:

Let

$$V \propto S^{\frac{1+b}{2}} \tag{35a}$$

or

$$S \propto V^{\frac{2}{1+b}} \tag{35b}$$

For turbulent flow near rough boundaries, b = 0 , therefore S is proportional to V^2. In case of b is greater than zero, $s \propto V^{\beta<2}$, such as in the case of the Blasius equation. On the other hand, if b is less than zero, it means $S \propto V^{\beta>2}$, in other words, the head loss of the flow is proportional to the velocity to an exponent of greater than 2. Note that according to Lacey's regime theory (33), the mean velocity is a regime channel is proportional to the energy gradient to the one-third power.

By examination of Fig. 9 and Fig. 10, it can be concluded: (1) the Manning formula is applicable when the size of the bed material is 4 mm or greater, (2) the Blasius formula is applicable when the size of the bed material is 0.1 mm or smaller, (3) when the size of the bed material is 20 mm or greater, the effect of dune formation, if any, on the velocity is negligible and (4) the formation of dunes generates additional energy loss so that the energy loss is proportional to the velocity with an exponent normally greater than 2.

The coefficient C_a can be computed from Eq 32. However, such a method of determining C_a is very tedious. Instead the

C_a -curves were determined by substituting available data in Eq 31. Since C_a is not dimensionless, and its dimension depends upon the exponent of the hydraulic radius, therefore the C_a-value for the English system differs from that for the metric system (Figs. 11 and 12). The coefficient C_a should be a function of the properties of the sediment and the fluid. Under ordinary conditions of open channel flow, the density of the sediment is approximately constant. The shape of the sediment particle and the properties of water can also be considered approximately constant. Therefore, C_a is essentially a function of the sediment size alone. It should be noted that the temperature variation of the data was between 15°C and 30°C. It was found that the variation of C_a due to temperature change is less than that due to error of measurements. That the effect of the variation of viscosity of the water on the mean velocity is small has been shown to be true even in the Blasius formula. Therefore, the effect of the temperature on the discharge coefficient can be neglected for practical purposes.

Note that the effect of temperature is included in Eq 24, which has been reduced to Eq 31. However, the effect of temperature is neglected in the determination of C_a -curves. The explanation for this is that, if the viscosity factor is used to form a certain dimensionless parameter in the study of alluvial roughness, the inclusion of the effect of viscosity should be complete. On the other hand, for practical purposes, the effect of neglecting viscosity on the mean velocity may be small compared to errors of other sources.

Because the exponents for the hydraulic radius and the slope S depend upon the bed configuration, separate C_a -curves for plane bed and for dune bed are also necessary as shown both in Fig. 11 and in Fig. 12.

In the case of a plane bed, the discharge coefficient C_a decreases as the bed material size d increases except when d is smaller than 0.1 mm, then the C_a -value is essentially a constant equal to 287 in Fig. 11. This is the case where the Blasius equation is applicable, and where the discharge coefficient is independent of the height of the boundary roughness. Between d = 0.2 mm and d = 1 mm, the value of C_a decreases rapidly as the value of d increases. Between d = 4 mm and d = ~~100~~ 80 mm, the value of C_a can be written as

For English system $C_a = 112 - 30.5 \log_{10} d$ d in mm (36a)

for metric system $C_a = 75 - 21 \log_{10} d$ d in mm . (36b)

Equation 36 corresponds to the Strickler formula shown as Eq 16.

In the case of a dune bed, at the bed material size d equal to about 0.2 mm, the discharge coefficient is a minimum, which may be interpreted as that the effect of dunes on the discharge coefficient is the greatest at d = 0.2 mm; and at the size of less than 0.04 mm, the discharge coefficient is essentially constant at 21. The discharge coefficient C_a increases as the size of the bed material increases from .2 mm to 7 mm, which means that within this range of the bed material size the effect of dunes on the discharge coefficient decreases although the size of the bed material increases. The C_a -curve for dune bed coincides with that for plane bed at d = 20 mm, which means that for d equal to 20 mm or greater the effect of dune on the discharge coefficient is negligible.

It was pointed out earlier that the bed configuration after the beginning of motion can be classified as plane, ripples, dunes, bars, flat, and antidunes. The case of antidune is excluded entirely from this discussion because of insufficient data. The most important concept out of this research is that the flow over an alluvial bed is divided into two classes: that is, the flow with a plane bed, and the flow with a dune bed. The discharge coefficient C_a and the exponents x and y depend upon not only the bed material size but also the bed configuration, plane bed or dune bed.

In the case of ripple-bed, which is the transitional stage between plane bed and dune bed and in the case of sand bars, which follows the stage of dunes, the values of the exponents x and y for dune bed have been used in determining the C_a -value. The C_a -curve for ripples and sand bars is shown immediately above the C_a -curve for dunes. The two curves coincide with each other for the bed material sizes less than about 0.06 mm, and greater than about 1.6 mm, which means, the distinction between ripples, bars, and dunes as far as their effect on the discharge is concerned is nil for $d < 0.06$ mm and for $d > 1.6$ mm (It can be found from Fig. 1 that ripples will not form when d is greater than about 2 mm). That the C_a -curve for ripples and bars lies above that for dunes means that the resistance of ripples and sand bars is generally less than that of dunes. The minimum C_a -value for ripples and bars is about 18 at d equal to about 0.16 mm.

In the case of a flat bed, which follows the stage of sand bars, the values of the exponents x and y for plane bed have been used in determining the C_a -value which is shown immediately below the C_a -curve for plane bed. The C_a -value reaches essentially a constant of 256 when the bed material size is less than 0.1 mm. The C_a -curve of flat bed coincides with that of plane bed when the bed material size is equal to or greater than 20 mm. That the C_a -curve of the flat bed in general lies below that of plane bed indicates that the discharge coefficient of a flat bed is normally less than that of a plane bed. This means, for given bed material, depth of flow and slope of the channel, the discharge or velocity of a flow with flat bed is less than that of a flow with plane bed. Actually such an exact comparison of discharge is impossible since, according to Fig. 1, there is only one type of bed configuration for a given combination of the bed material size, the depth of flow and the slope of the channel.

It should be noted that in order to select the x and y exponents and the appropriate discharge coefficient, it is necessary to estimate the bed configuration, which requires the use of Fig. 1. The following is an illustration to compute the mean velocity of flow:

Given:

$$d = 0.483 \text{ mm}$$

$$S = 0.001$$

$$R_b = 0.397 \text{ ft}$$

$$t = 60.5^0 \text{ F}$$

Required: V

Computation procedure:

$$V_* = \sqrt{gR_b s} = 0.1131 \text{ fps}$$

assume the sediment is spherical, $w = 0.234$ fps

and $\frac{wd}{\nu} = 30.9$

therefore $\frac{V_*}{w} = 0.483$

From Fig. 1 it is estimated that the bed would be a dune-bed. Therefore the exponents and coefficient for a dune-bed are chosen:

From Fig. 11 $C_a = 17$

From Fig. 9 $x = 0.507$

From Fig. 10 $y = 0.328$

Substitute these values in Eq 31.

$$V = 17 \times 0.397^{0.507} \times 0.001^{0.328} = 1.19 \text{ fpS} .$$

The measured velocity was 1.11 ft per second.

The data computed in this manner are shown in Figs. 13 and 14. In Fig. 13 which is for a plane bed, 83 per cent of the data are within 10 per cent of scatter. None of the computed velocity exceeds 20 per cent of deviation from the measured value. In Fig. 14, which is for dune bed, 70 per cent of the data are within 10 per cent of scatter and only 1 per cent of the data exceeds 20 per cent of deviation from the measured value. In general the average error is about 10 per cent or less. In view of the fact that it is very difficult to obtain accurate data of flow in alluvial channels, such as measuring the depth of flow and the energy slope, an average discrepancy of 10 per cent in computing the velocity can be considered acceptable for engineering purposes.

The results presented here are primarily for two-dimensional flow. Any effect of a side wall must be eliminated by using standard procedures. This involves a method of trail and error since R_b cannot be computed without knowing the mean velocity. In order to overcome this difficulty, both the total discharge and the depth of flow must first be assumed. The mean velocity can be found according to the equation of continuity, then R_b can be computed. By using R_b, the slope of the channel, and the mean size of the bed-material, the mean velocity can be checked according to Eq 31. This method should be used by repeated trials until the result is consistant.

Eq 31 is suitable for steady, uniform flow. However, this is not the case for most of the natural streams. In the case of natural streams, the discharge coefficient probably has to be modified to suit the field condition. The exponents for hydraulic radius and slope probably can be the same as shown in Figs. 11 and 12 -- at least as a first approximation.

VI. SUGGESTIONS FOR FUTURE RESEARCH

As pointed out earlier, it is impossible at the present to find a theoretical solution of the mean velocity of alluvial streams, therefore the authors have proposed certain methods of empirical correlation. In so doing it was necessary to make some assumptions for simplification. However, in order to understand the problem thoroughly so that the final solution of mean velocity of alluvial streams can be obtained, some additional research work definitely is needed. The needed research is almost unlimited. The following suggestions are only those which are directly related to the present approach.

1. The information on bed configuration is very important in order to apply the discharge formula properly, the classification of bed configuration needs to be defined more accurately.
2. In this study the effect of sediment shape was not considered. All sediment particles were assumed to be spherical. In order to improve the accuracy of the method, it is necessary to determine the effect of particle shape on the mean velocity.
3. The effect of the mixture (size gradation) has not been investigated thoroughly, further research is needed to determine its effect on the mean velocity. The accuracy of the authors" method depends considerably upon the size of the bed-material.
4. In order to improve the accuracy of the formula, the C_a -

curves shown in Figs. 11 and 12 should be classified more accurately according to the bed configurations.

5. It is desirable to express the exponents x and y, and the discharge coefficient C_a as a function of a certain dimensionless parameter or parameters. Further research to improve Figs. 6, 9, 10, 11 and 12 is needed.

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REFERENCES

1. Hwang, S. Y. "Analytical Study of the Roughness of Alluvial channels," unpublished M. S. thesis. Civil Engineering Dept., Colorado State University, Fort Collins, Colorado. 1958.

2. Rouse, H. and S. Ince. "History of Hydraulics," Iowa Institute of Hydraulic Research, Iowa City, Iowa. 1957. pp. 269.

3. Manning, R. "On the Flow of Water in Open Channels and Pipes," Trans. Inst. C. E., Ireland, Vol. 20, 1890.

4. Kolupaila, S. "Hidraulika (Hydraulics), in Lithuanian language, Kempten, Bavaria. 1947. pp. 288.

5. Prandtl, L. "Turbulent Flow" (Überdie ausgebildete Turbulenz). Proc. of 2nd International Congress of Appl. Mech., Zürich, 1926, NACA TM No. 1231, 1949.

6. Kármán, T. Von. "Mechanical Similitude and Turbulence" (Mechanische Ahnlichkeit and Turbulenz), Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Fachgruppe I (Mathematik), No. 5, 1930, NACA TM No. 61). March 1931.

7. Rouse, H. Elementary Mechanics of Fluids, John Wiley and Sons, New York. 1946.

8. Nikuradse, J. "Laws of Flow in Rough Pipes" (Stromungsgestze in rauhen Rohren). Verein Deutscher Ingenieure, Forschungsheft 361, Supplement to Forschung auf dem Gebiete des Ingenieur wesens, Series B, Vol. 4. July - August 1933, NACA TM No. 1292. November 1950.

9. Keulegan, Garbis H., "Laws of Turbulent Flow in Open Channels." U. S. Department of Commerce, Bureau of Standard Research paper RP 1151. Journal of Research of the National Bureau of Standards, Vol. 21, December 1938.

10. Powell, R. W. "Resistance to Flow in Rough Channels." Trans AGU, Vol. 31, No. 4. 1950.

11. Robinson, A. R. and M. L. Albertson. "Artificial Roughness in Open Channels," Trans. AGU, Vol. 33, p. 881, 1952.

12. Sayre, W. W. "Artificial Roughness Patterns in Open Channels", unpublished M. S. thesis, Civil Engineering Dept., Colorado State University, Fort Collins, Colorado. 1957.

13. Johnson, J. W. "Tectangular Artificial Roughness in Open Channels," Trans. AGU, part V, 1944.

14. Millikan, C. B. "A Critical Discussion of Turbulent Flow in Channels and Circular Tubes." Proc. 5th International Congress Applied Mechanics, 1939. pp. 386-392.

15. Schlichting, H. Boundary Layer Theory, McGraw-Hill Co., New York. 1955.

16. Blasius, H. Das Ahnlichkeitsgesetz bei Reibungsvorgangen in Flussigkeiten (The law of similarity for friction in fluids). Verein Deutscher Ingenieure, Forschungsheft p. 131. 1913.

17. Clauser, F. H. "The Turbulent Boundary Layer, " Adv. in Appl. Mech. IV, Academic Press, New York. 1956. pp. 51.

18. Einstein, H. A. "The Bed-load Function for Sediment Transportation in Open Channel Flows. United States Department of Agriculture, Techn. Bull. no. 1026. 1950.

19. Kalinske, A. A. "Movement of Sediment as Bed Load in Rivers." Trans. AGU. 1947.

20. Laursen, E. M. "The Total Sediment Load of Streams." Journal of Hyd. Div., Vol. 84, No. HY1, pt. 1, Proc. ASCE, February 1958.

21. Strickler, A. Beiträge zur Frage der Geschwindigheits-formel und der Rauhigkeitszahlen Für Ströme, Kanäle und geschlossene Leitungen Edig. Wasserwirtschaftsamt Berne, Mitteilung, 1923.

22. Einstein, H. A. and N. L. Barbarossa. "River Channel Roughness," Trans. A.S.C.E., Vol. 117, 1952, pp. 1121.

23. Vanoni, V. A. and N. H. Brooks. "Laboratory Studies of the Roughness and Suspended Load of Alluvial Streams. Rept. No. E-68, California Institute of Technology, Pasadena, California. December 1957.

24. Exner, F. M. "Über die Wechselwirkung Zwischen Wasser und Geschiebe in Flüssen." Sitzenberichte, Academie der Wissenshaften, Vienna, Vols. 3-4, 1925.

25. Liu, H. K. Discussion on "Mechanics of Sediment Ripple Formation," Proc. 1832, Journal of Hyd. Division, Vol. 84, No. HY5, October 1958. pp. 1832-5.

26. Shields A. Anwendung der Ähnlichkeits mechanik und der Turbulenz Forschung auf die Geschieblewegung Mitteilungen der Preussischen Versuchsanstalt für Wasserbau und Schiffbau Heft 26, Berlin, 1936.

27. Liu, H. K. "Mechanics of Sediment-Ripple Formation." Journal of Hyd. Div., Vol. 83, HY2, Proc. A.S.C.E. April, 1957.

28. Albertson, M. L., D. B. Simons, E. V. Richardson. Discussion on "Mechanics of Sediment-Ripple Formation" by H. K. Liu, Vol. 84, HY1, Journal of Hyd. Div. Proc. A.S.C.E., February 1958.

29. U. S. Waterway Experimental Station. "Studies of River Bed-Materials and their Movement, with Special Reference to the Lower Mississippi River." Paper 17, January 1935.

30. Einstein, H. A. "Formula for the Transportation of Bed-Load Movement." Trans. A.S.C.E., Vol. 107, 1942.

31. Liu, H. K. "Ripple Formation and Its Relation to Bed-Load Movement," unpublished Ph.D. Thesis, 1953, Civil Engineering Dept., University of Minneapolis, Minneapolis, Minnesota.

32. Chang, Y. L. "Laboratory Investigations of Flume Traction and Transportation." Trans. A.S.C.E., Vol. 104, 1939.

33. Lacey, G. Discussion on "A Concept of the Regime Theory" by N. Chien. Trans. A.S.C.E., Vol. 122, 1957. pp. 794.

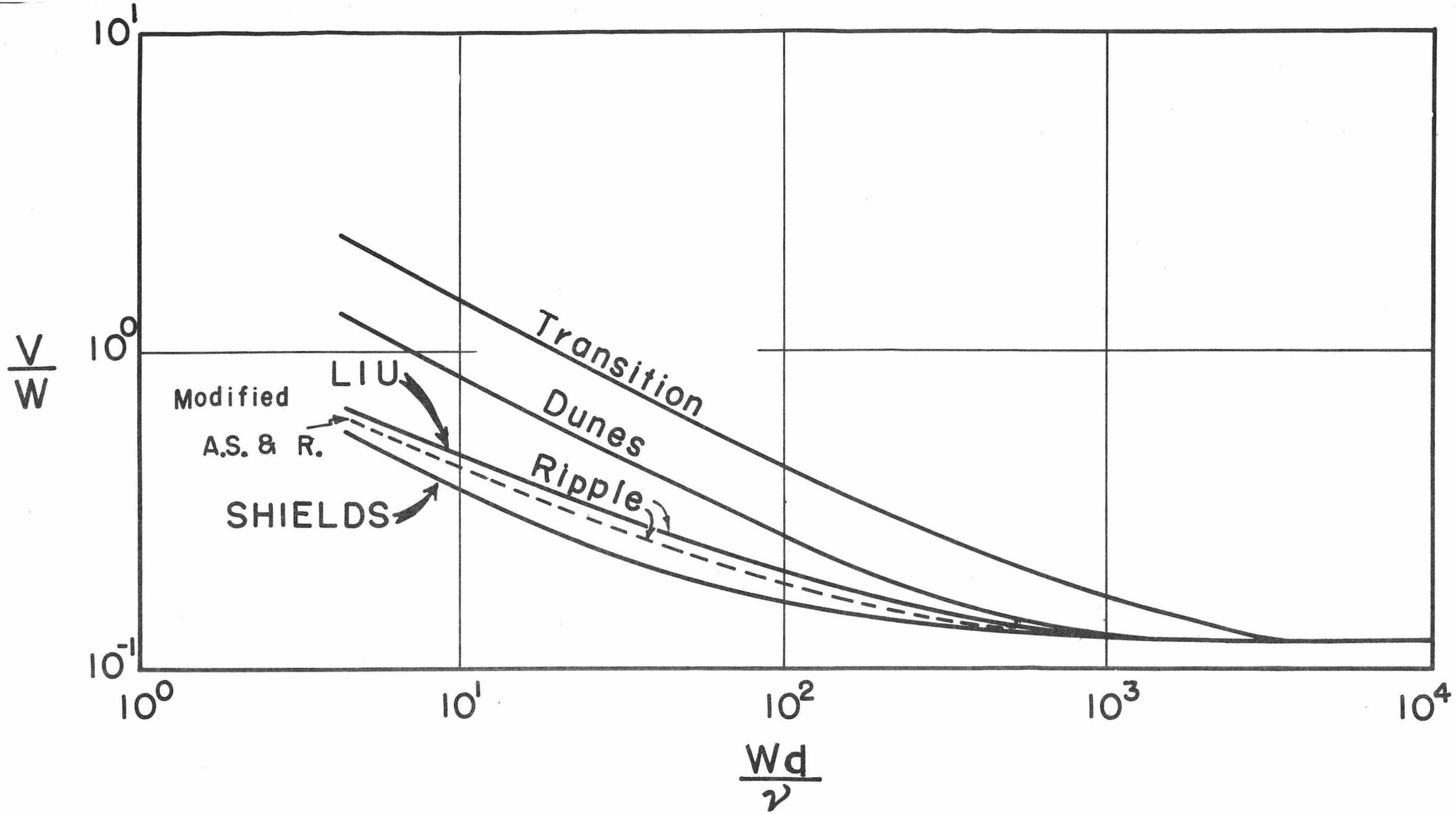

FIG. 1 CLASSIFICATION OF THE CONFIGURATION OF ALLUVIAL BED

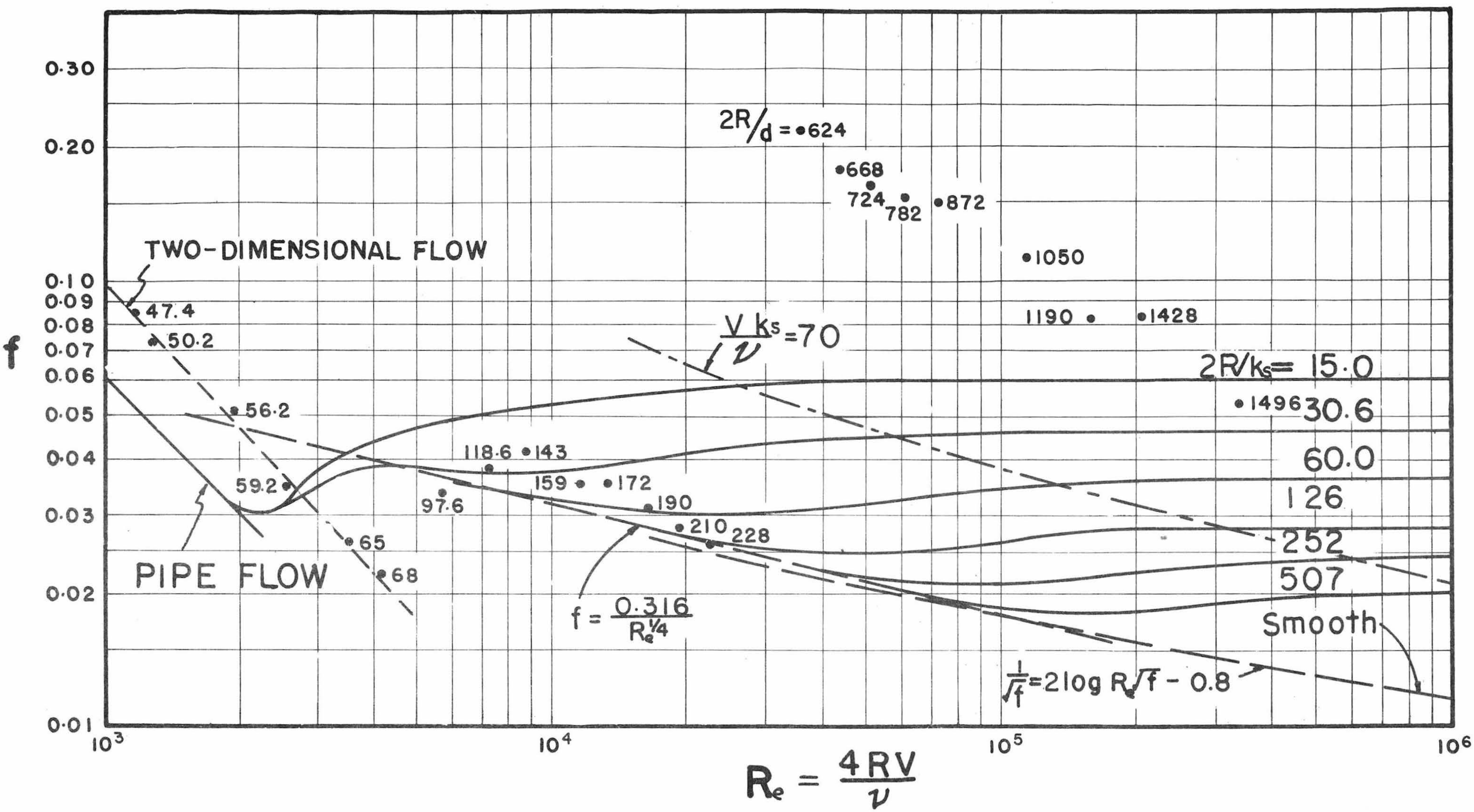

FIG. 2 VARIATION OF f WITH $\frac{4RV}{\nu}$ FOR ALLUVIAL BED

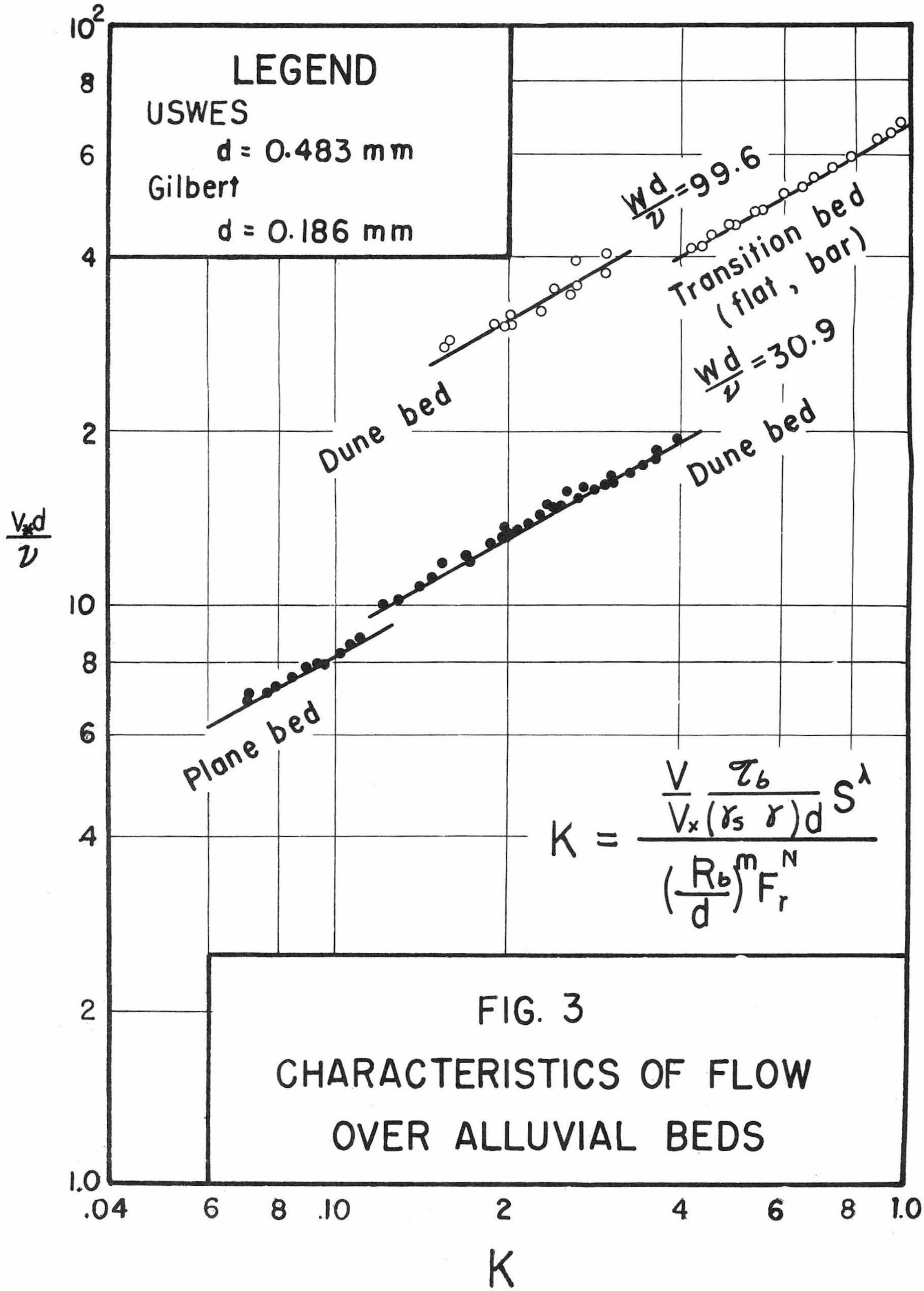

FIG. 3

CHARACTERISTICS OF FLOW OVER ALLUVIAL BEDS

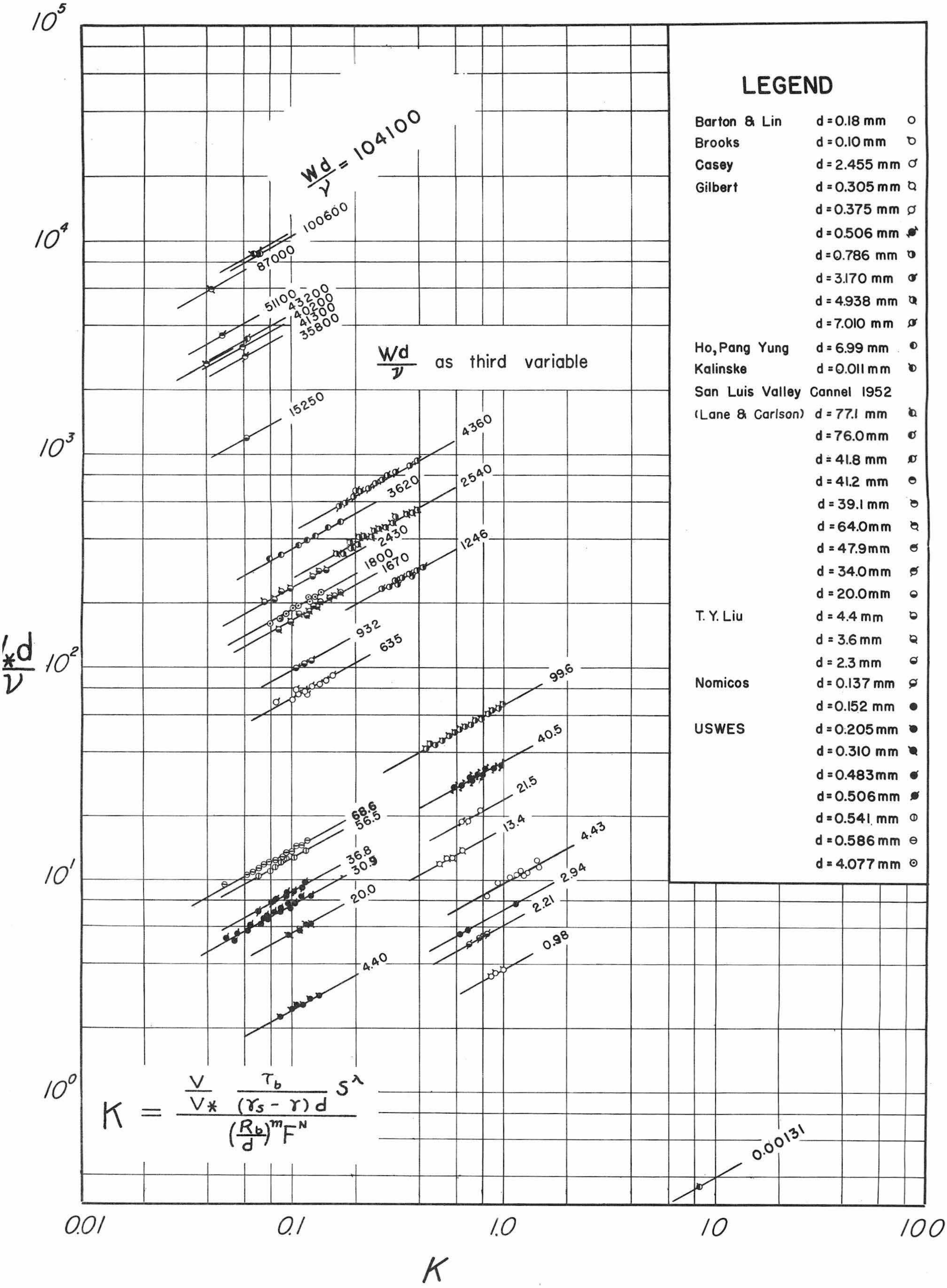

FIG. 4 CHARACTERISTICS OF FLOW OVER ALLUVIAL PLANE BED

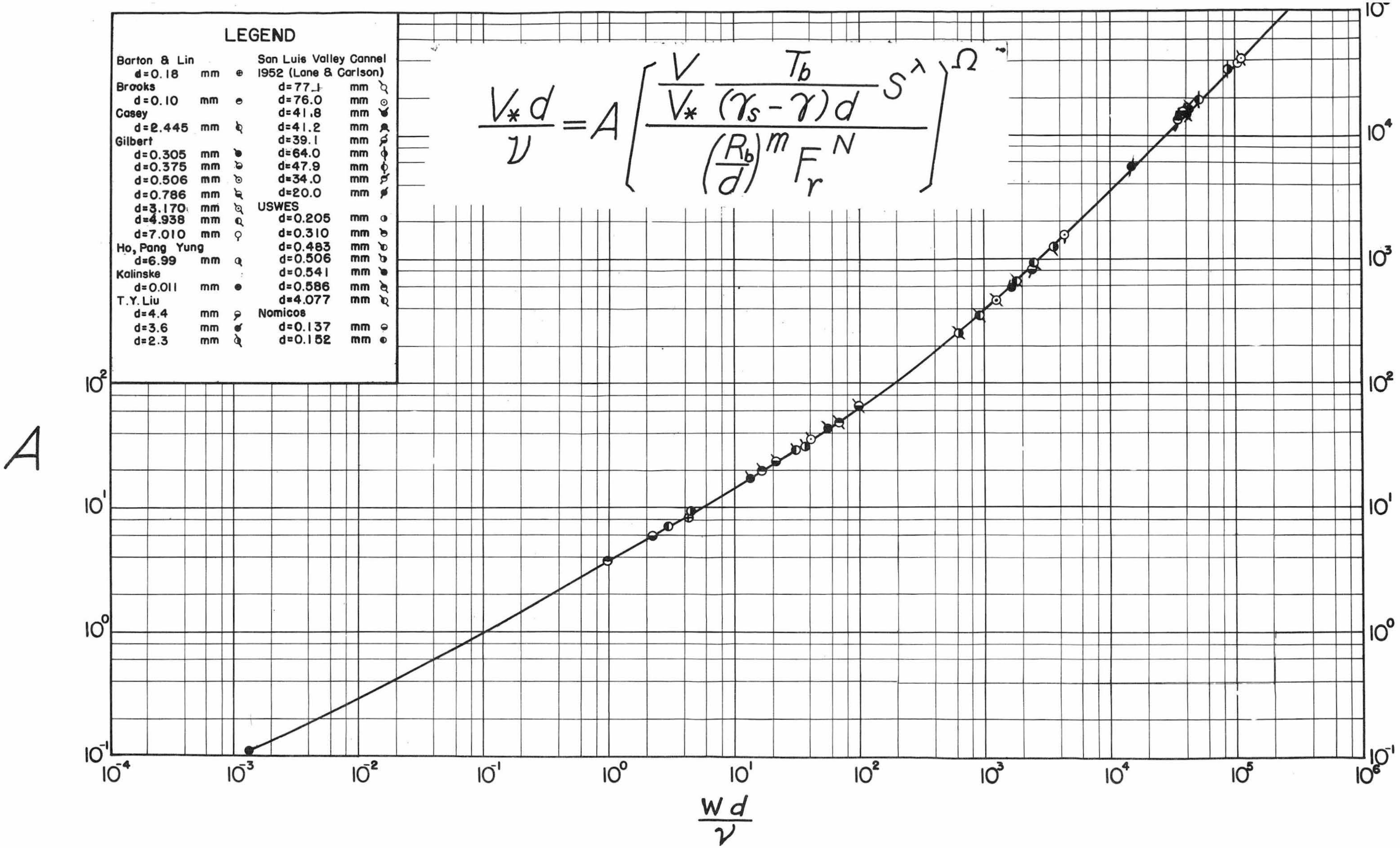

FIG. 5 VARIATION OF A WITH $\frac{wd}{\nu}$ FOR ALLUVIAL PLANE BED

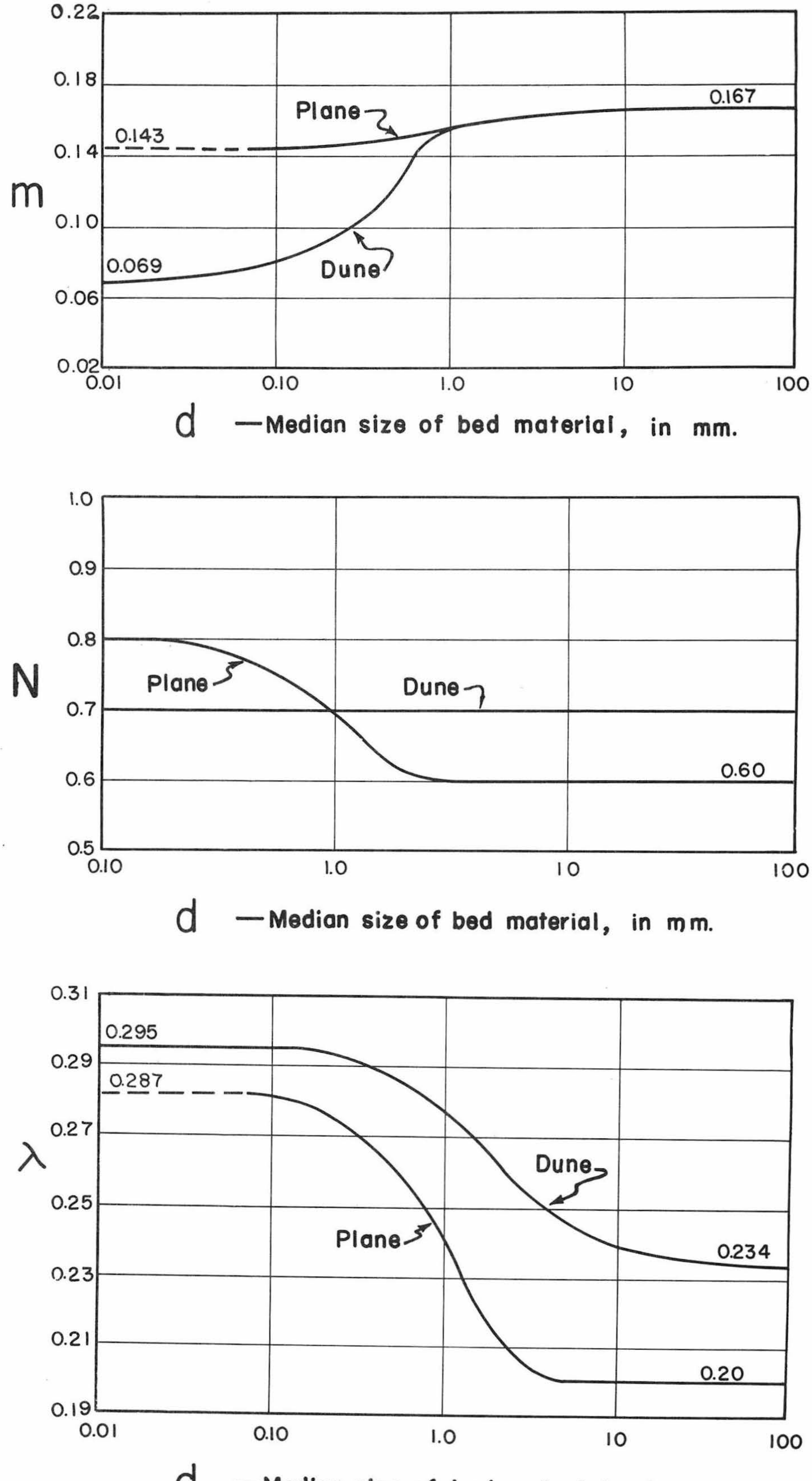

FIG. 6 VARIATION OF λ, m, AND N WITH d FOR FLOW OVER ALLUVIAL BED

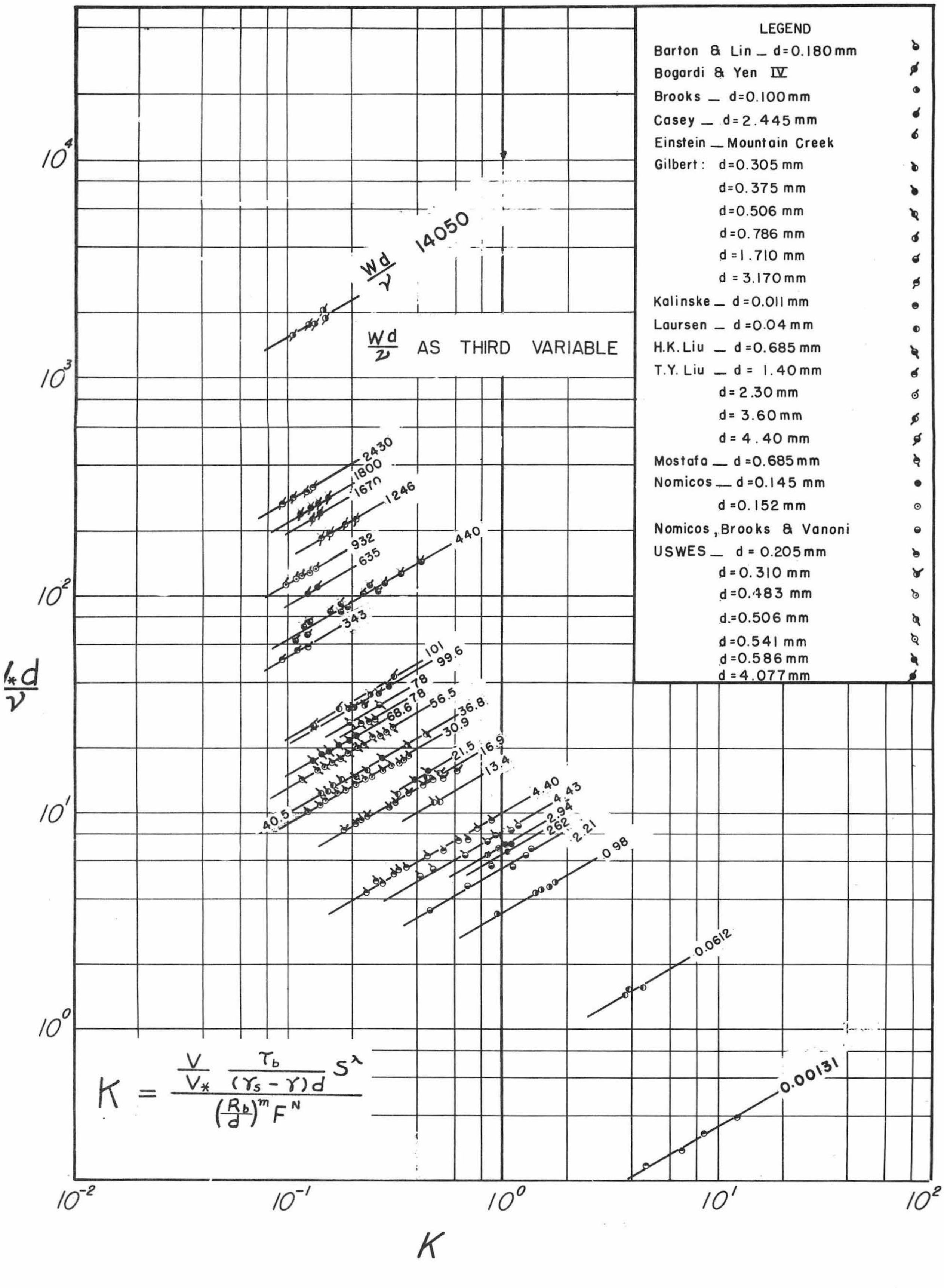

FIG. 7 CHARACTERISTICS OF FLOW OVER ALLUVIAL DUNE BED

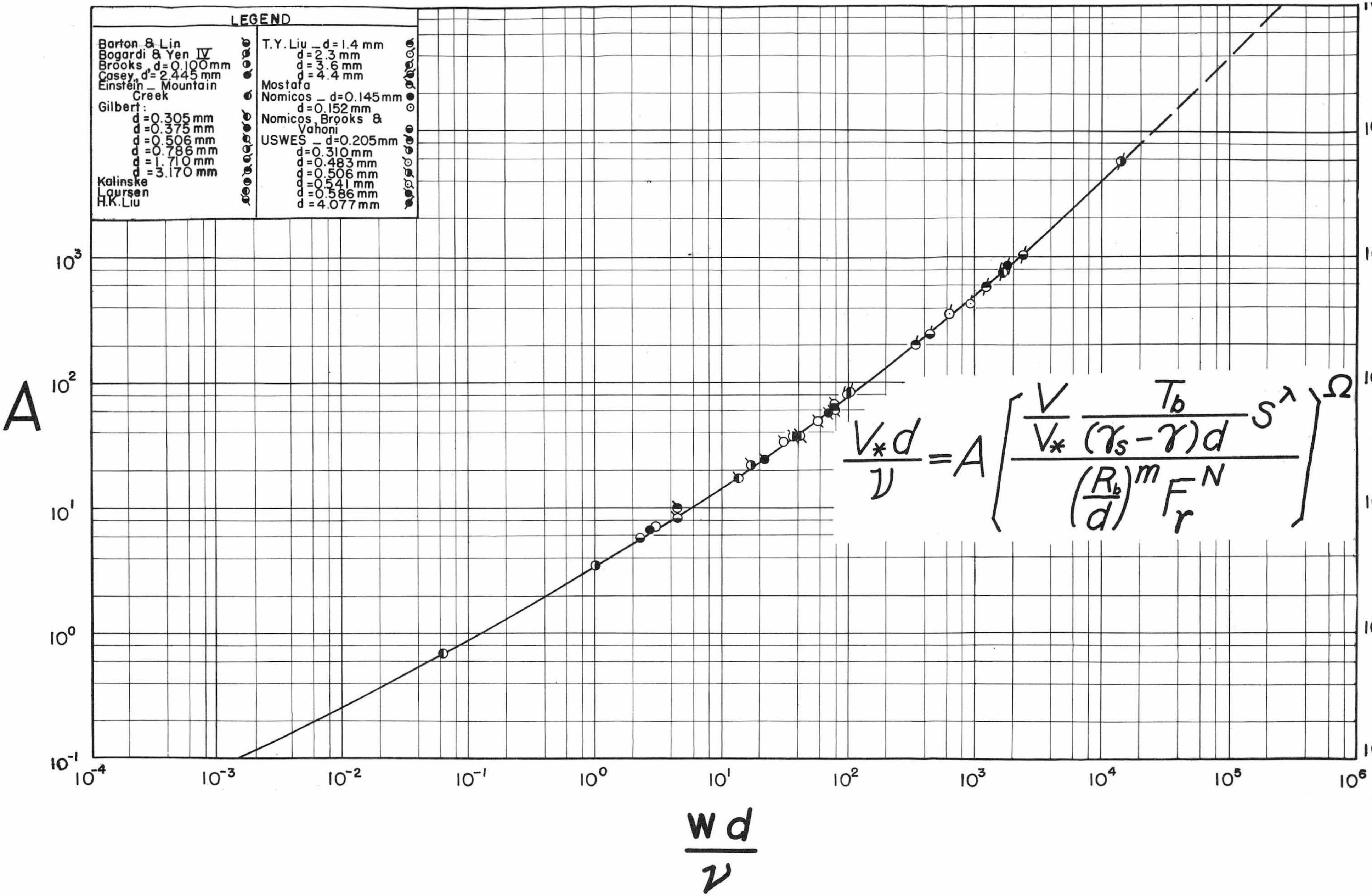

FIG. 8 VARIATION OF A WITH $\frac{Wd}{\nu}$ FOR ALLUVIAL DUNE BED

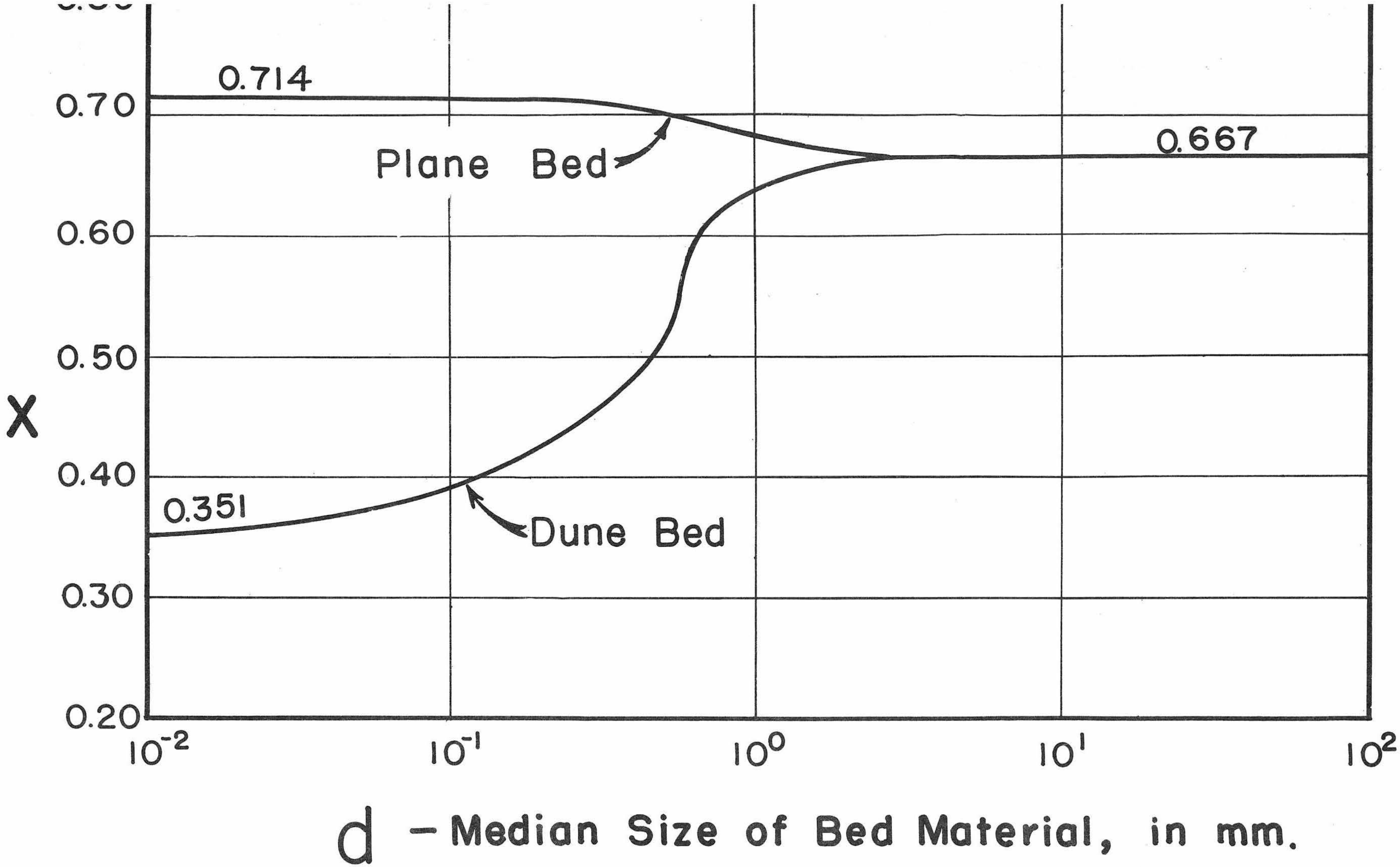

FIG. 9 VARIATION OF X WITH d FOR FLOW OVER ALLUVIAL BED

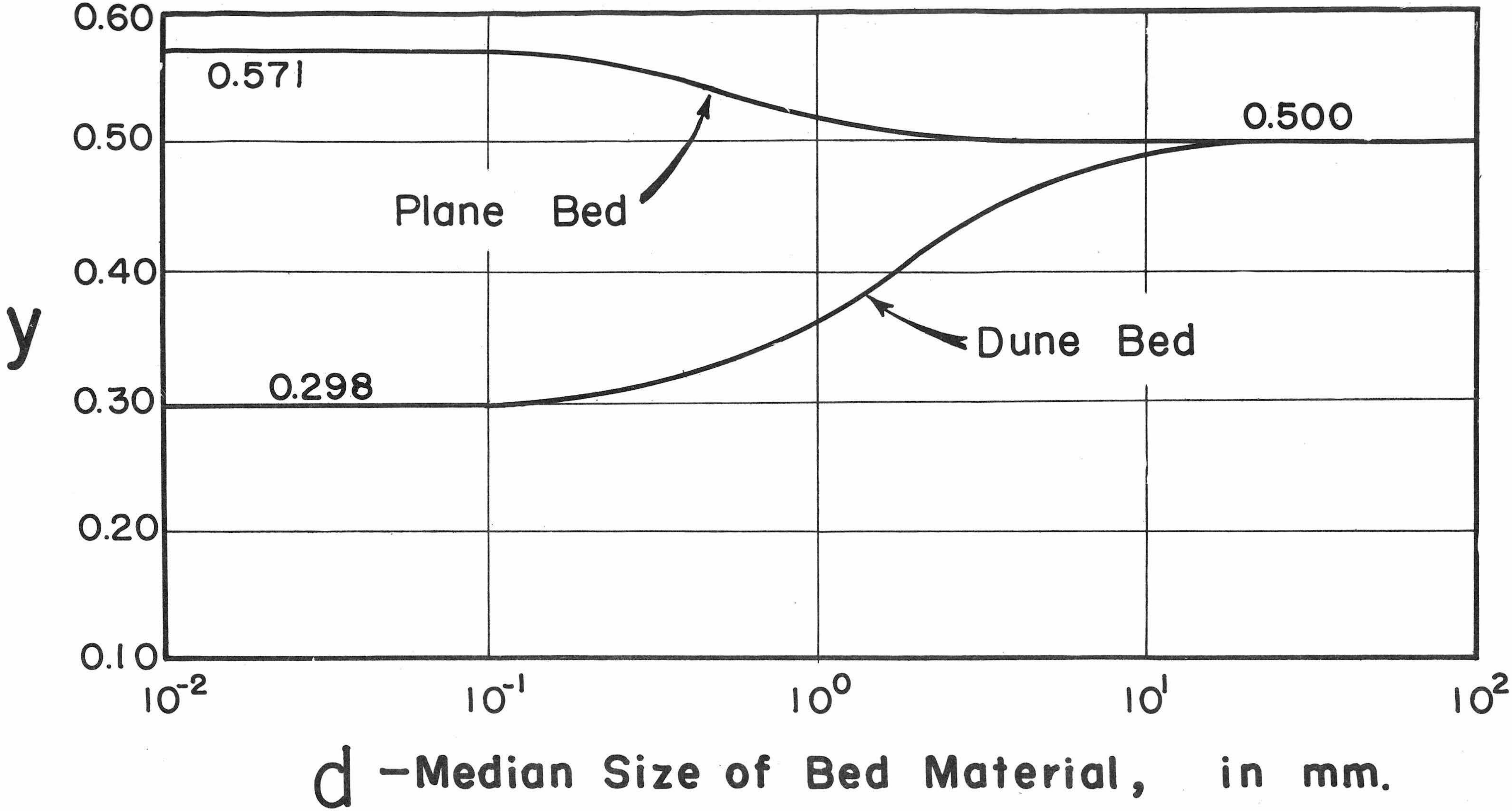

FIG. 10 VARIATION OF Y WITH d FOR FLOW OVER ALLUVIAL BED

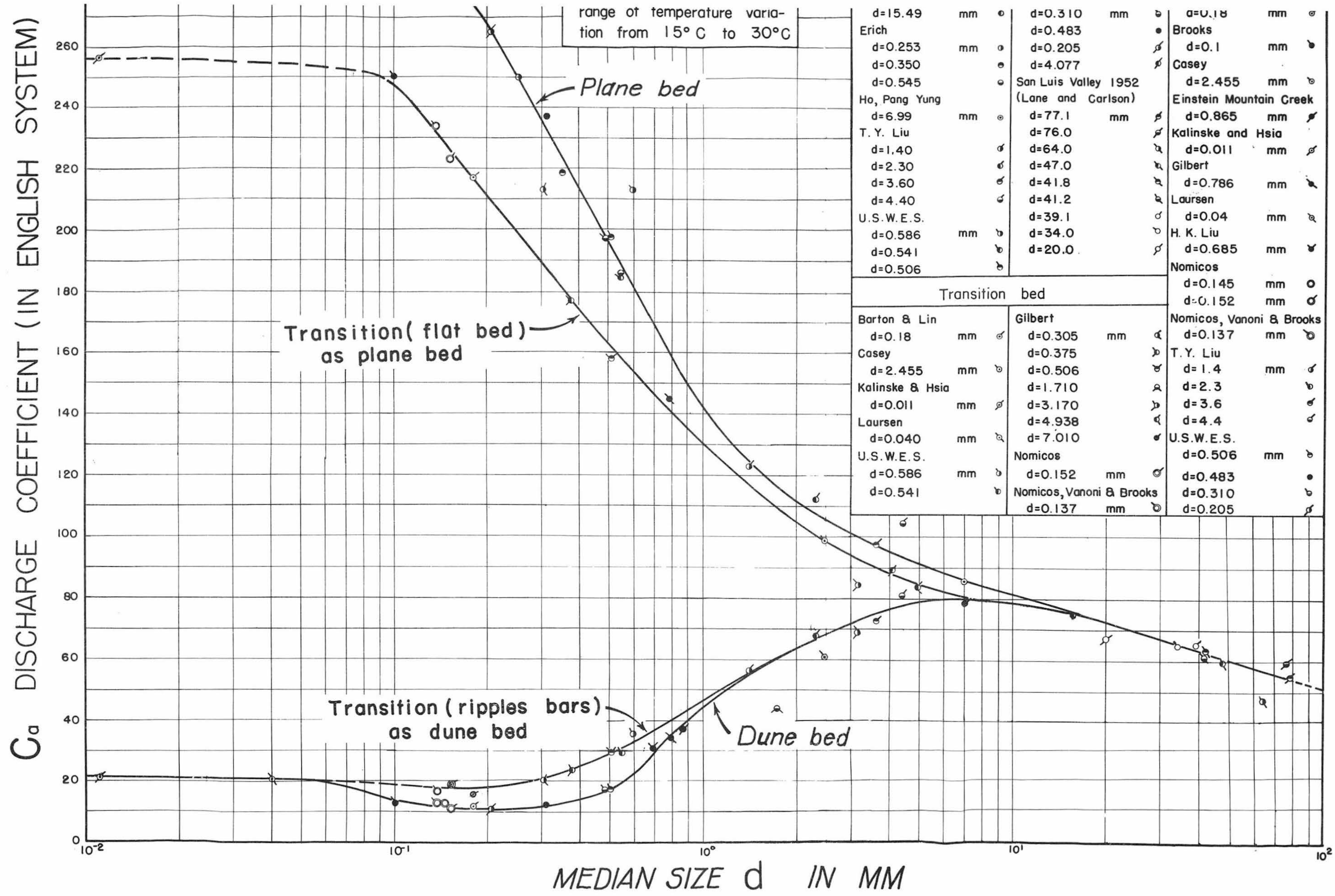

FIG. 11 VARIATION OF C_a (IN ENGLISH SYSTEM) WITH d FOR FLOW OVER ALLUVIAL BED

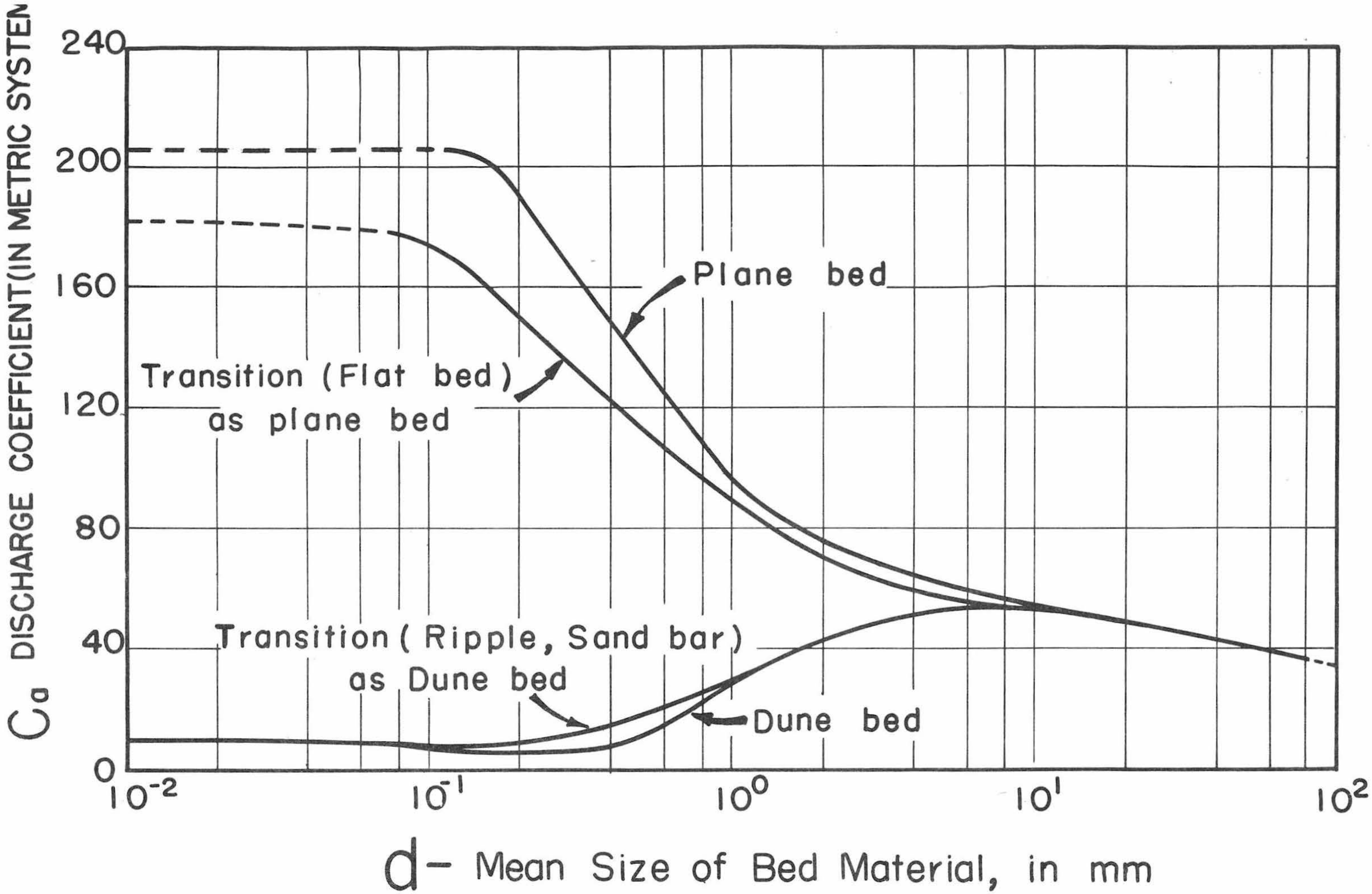

FIG. 12 VARIATION OF C_a (IN METRIC SYSTEM) WITH d FOR FLOW OVER ALLUVIAL BED

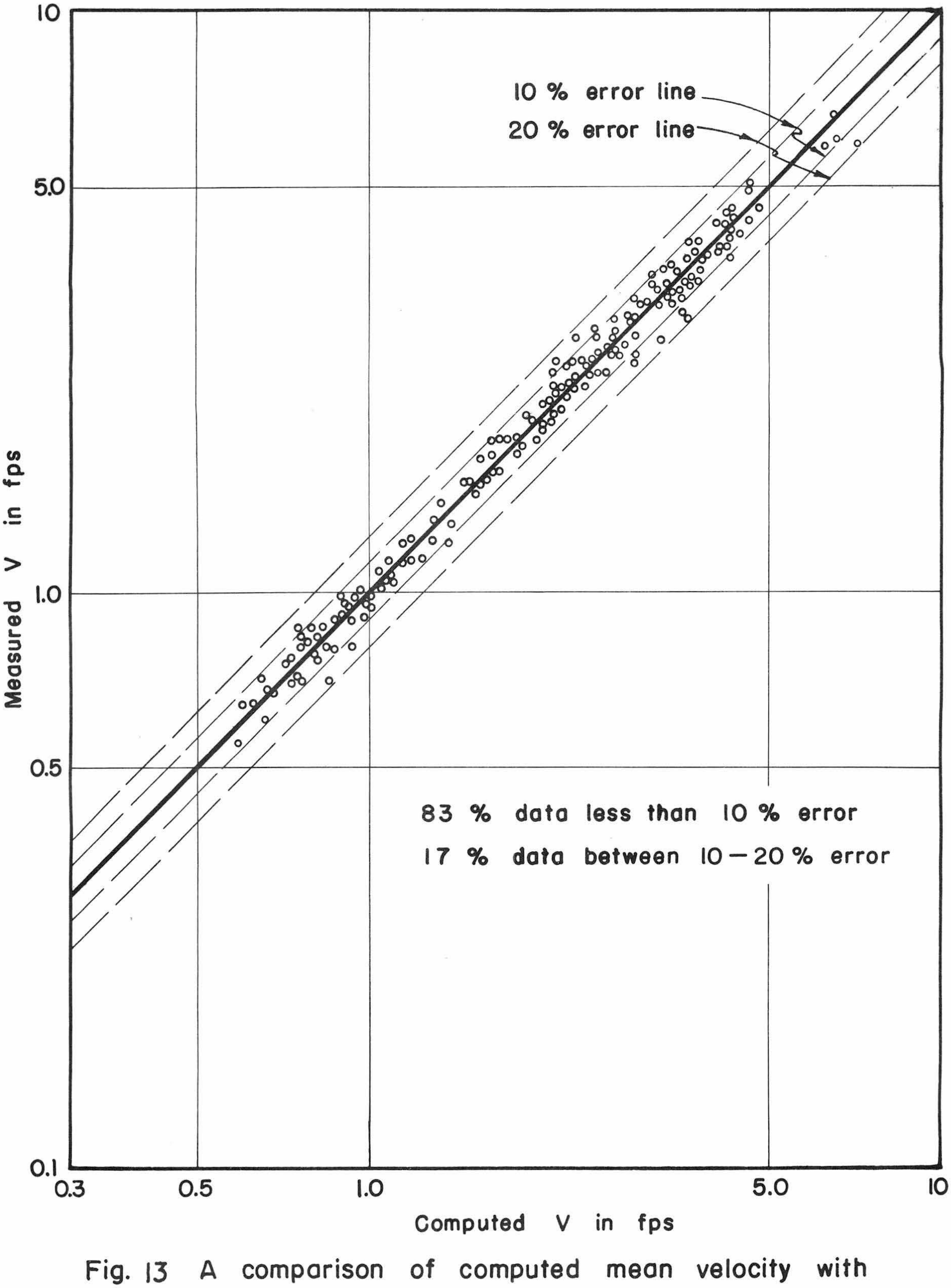

Fig. 13 A comparison of computed mean velocity with measured mean velocity for alluvial plane bed

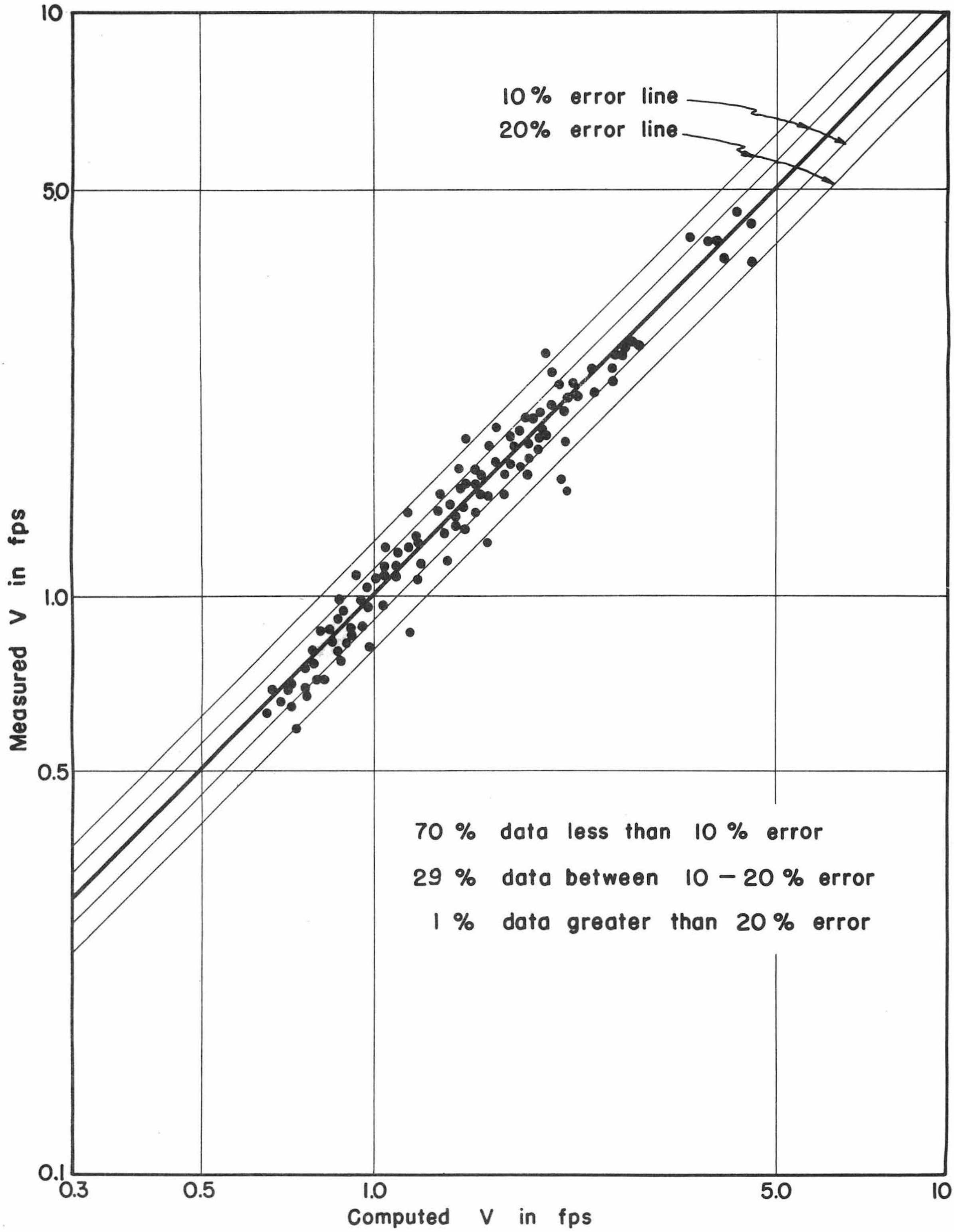

Fig. 14 A comparison of computed mean velocity with measured mean velocity for alluvial dune bed