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Three-membered Domains for Aristotle's Syllogistic[1]

Abstract. The paper shows that for any invalid polysyllogism there is a procedure for constructing a model with a domain with exactly three members and an interpretation that assigns non-empty, non-universal subsets of the domain to terms such that the model invalidates the polysyllogism.

Given the traditional definition of the semantic logical consequence relation for Aristotle's syllogistic, where terms designate non-empty subsets of a domain, we can exhibit x and y, where x is a set of sentences and y is a sentence, such that a domain with *at least three* members is required to show that y is not a logical consequence of x. (For example, it takes a three-membered domain to show that "Some a are c" is not a logical consequence of "No b are c" and "No a are b.") This paper shows that if y is not a logical consequence of x then, no matter how many terms are involved, we can show this using a domain with *no more than three* memberes. Moreover, thanks to correspondence with T.J. Smiley, it is shown that only *proper* subsets of three-membered domains are needed to interpret terms.

The syntax of the syllogistic:

Terms: $a_1, a_2, \ldots$.

Primitive operators: A, E, I, O

Defined operators: A', E', I', O' ($A'xy = Ayx$, $E'xy = Eyx$, $I'xy = Iyx$, and $O'xy = Oyx$. A, E, I, and O are *basic operators*. A, I, A', and I' are *positive operators*, and the others are *negative operators*.

Sentences: If x and y are terms and Q is an operator then Qxy is a *sentence*, and no other expressions are sentences. (x is the subject term and y the predicate term of Qxy.)

Next, we give the semantics for the syllogistic. $\langle D, \mathcal{J}\rangle$ is a *model* iff D is a non-empty set and $\mathcal{J}$ is a function whose domain is the set of terms and sentences, where $\mathcal{J}$ meets these conditions, *conditions for a model*: i) If x is a term then $\mathcal{J}(x)$ is a non-empty, non-universal subset of D, and ii) If x is and y are terms then $\mathcal{J}(Axy) = t$ iff $\mathcal{J}(x) \subseteq \mathcal{J}(y)$, $\mathcal{J}(Exy) = t$ iff $\mathcal{J}(x) \cap \mathcal{J}(y) = \emptyset$,

[1] I am grateful to T. J. Smiley and an anonymous referee for comments that improved this paper.

$\mathcal{J}(Ixy) = t$ iff$\mathcal{J}(Exy) = f$, and $\mathcal{J}(Oxy) = t$ iff $\mathcal{J}(Axy) = f$. If $x_1, \ldots, x_n$ are sentences then x_n *is a logical consequence of* $x_1, \ldots, x_{n-1}$ ($x_1, \ldots, x_{n-1} \models x_n$) iff there is no model $\langle D, \mathcal{J} \rangle$ such that$\mathcal{J}(x_i) = t$ $(1 \leq i < n)$ and $\mathcal{J}(x_n) = f$.

DEFINITION. $\langle x_1, \ldots, x_n \rangle$ (where $n \geq 1$) is an *Aristotelian chain* iff each x_i $(1 \leq i \leq n)$ is a sentence, any term that occurs in the members of the sequence occurs exactly twice, a term that occurs in x_i (for $1 \leq i \leq n$) occurs in x_{i+1}. (We are adding modulo n. So, a term that occurs in x_n occurs in x_1.)

THEOREM. If $\langle x_1, \ldots, x_n \rangle$ is an Aristotelian chain then $x_1, \ldots, x_{n-1} \models x_n$ iff there is no model $\langle D, \mathcal{J} \rangle$, where D has three members, such that $\mathcal{J}(x_i) = t$ $(1 \leq i \leq n)$ and $\mathcal{J}(x_n) = f$.

PROOF. (Only if) Use the definition of logical consequence.

(If) Given familiar reasoning about antilogisms, we only need to show that if there is a model $\langle D, \mathcal{J} \rangle$ such that $\mathcal{J}$ assigns t to each member of an Aristotelian chain AC ($\mathcal{J}$ *satisfies* AC) then there is a model $\langle D', \mathcal{J}' \rangle$ such that D' has three members and $\mathcal{J}'$ satisfies AC. We show this by establishing four lemmas, which use the notion of a normal chain.

DEFINITION. An Aristotelian chain is a *normal chain* iff for any two consecutive sentences in a chain the predicate of the earlier one is the subject of the later one (and the predicate of the last is the subject of the first). (In a normal chain $\langle x_1, \ldots, x_n \rangle$ x_i and x_{i+1} are consecutive sentences. In particular, x_n and x_1 are consecutive sentences.)

LEMMA 1. *Any Aristotelian chain can be rewritten as a normal chain.*

PROOF. Straightforward.

(So, for example, $\langle Aa_2a_1, Oa_2a_3, Ea_1a_3 \rangle$ can be rewritten as $\langle A'a_1a_2, Oa_2a_3, E'a_3a_1 \rangle$.)

DEFINITION. A normal chain X *immediately reduces* to chain Y iff Y can be formed by replacing consecutive sentences 'Q_1ab, Q_2bc' in X by 'Q_3ac', where $Q_1Q_2Q_3 = AAA$, AE(or E')E, $AO'O'$, E(or E')AO', E(or E')I(or I')O', E(or E')$A'E$, I(or I')AI, I(or I')E(or E')O, $OA'O'$, $A'AI$, $A'E$(or E')O, $A'I$(or I')I, $A'OO$, $A'A'A'$, $O'AO'$. (The order of the "premises" matters. These "reduction rules" may be arranged as follows:

$$\frac{A \;\; A}{A} \quad \frac{A \;\; E}{E} \quad \frac{A \;\; E'}{E} \quad \frac{A \;\; O'}{O'} \quad \text{etc.)}$$

A normal chain *reduces* to chain Y iff there is a sequence of chains begining with X and ending with Y such that each member of the sequence (other than the last) immediately reduces to its successor. (Note that if a normal chain X reduces to chain Y then Y is a normal chain.)

(So, for example, $\langle A'a_1a_2, Ea_2a_3, Ia_3a_4, Oa_4a_1\rangle$ is a normal chain, and it immediately reduces to $\langle A'a_1a_2, O'a_2a_4, Oa_4a_1\rangle$, and it reduces to $\langle Oa_4a_2, O'a_2a_4\rangle$. Note also that the last chain is *irreducible*, given that it does not reduce to any chain.)

LEMMA 2. *The normal chains that reduce to a chain consisting of a single sentence with a negative operator are unsatisfiable.*

PROOF. For each normal chain consisting of a single negative sentence we give the chains which are reducible to it. Each of these chains is unsatisfiable given Theorem 2 in T. J. Smiley's 'What is a syllogism?' (*Journal of Philosophical Logic*, vol. 2, 1973, pp.136-154). Eaa (or $E'aa$): $Aa - b$, Ebc (or $E'cb$), $A'c - a$ (where '$Ax - y$' either designates nothing or a string of sentences $Axw, \ldots, Azy$, and where '$A'x - y$' either designates nothing or a string of sentences $A'xw, \ldots, A'zy$). Oaa: i) $A'a - b$, Obc, $A'c - a$, ii) $A'a - b$, $Ab - c$, Ecd (or $E'cd$), $A'd - a$, or iii) $A'a - b$, Ibc (or $I'bc$), $Ac - d$, Ede (or $E'de$), $A'e - a$. $O'aa$: i) $Aa - b$, $O'bc$, $Ac - a$, ii) $Aa - b$, Ebc (or $E'bc$), $A'c - d$, $Ad - a$, or iii) $Aa - b$, Ebc (or $E'bc$), $A'c - d$, Ide (or $I'de$), $Ae - a$. □

LEMMA 3. *Irreducible normal chains that are not chains consisting of a single sentence with a negative operator are satisfiable in a domain consisting of no more than three memberes.*

PROOF. We consider three cases, determined by the size of the irreducible normal chain. Case 1: A single sentence with a positive operator. Let $D = \{1,2,3\}$. The sentence has form Aaa, Iaa, $A'aa$, or $I'aa$. For each term x let $\mathcal{J}(x) = \{1\}$. Case 2: A pair of sentences. Then the operators of the sentences are: AO,EE, EO, EO',II, IO, IO', OO, OO', $A'O'$, or $O'O'$. Let $D = \{1,2,3\}$. If the sentences are Aab and Oba, let $\mathcal{J}(a) = \{1\}$, $\mathcal{J}(b) = \{1,2\}$, and let $\mathcal{J}(x) = \{1\}$ if $x \neq a$ or b. Desirable models with domain D for the other possible sentence pairs are easy to construct and will not be given. Case 3: An n-tuple of sentences where $n > 2$. Let the n-tuple be $\langle Q_1b_1b_2, Q_2b_2b_3, \ldots, Q_nb_nb_1\rangle$. The strategy is first to associate a three-rowed, n-columned matrix **M** with this sequence and then associate a domain (the rows of the matrix) and an interpretation $\mathcal{J}_{\mathrm{M}}$ with this matrix, defined as follows:

$$\mathcal{J}_{\mathrm{M}}(x) = \begin{cases} \text{the set of rows with } b_i \text{ as a member, if } x = b_1, \ldots \text{ or } b_n \\ \mathcal{J}_{\mathrm{M}}(b_1), \text{ if } x \neq b_1, \ldots \text{ and } b_n \end{cases}$$

The matrices will be defined so that $\mathcal{J}_{\mathrm{M}}$ meets condition i) for a model and so that $\mathcal{J}_{\mathrm{M}}$ assigns t to each sentence in the chain. And we stipulate that $\mathcal{J}_{\mathrm{M}}$ meets conditions ii) and iii) for a model. So, for example, the sequence $\langle Ea_1a_2, Ea_2a_3, Oa_3a_4, E'a_4a_1\rangle$ will have this associated matrix, **M**, given the full procedure specified below:

$$\begin{vmatrix} a_1 & \emptyset & a_3 & \emptyset \\ \emptyset & a_2 & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & a_4 \end{vmatrix}$$

Note that $\mathcal{J}_{\mathrm{M}}$ assigns t to each sentence in the chain. ($\mathcal{J}_{\mathrm{M}}(a_1)$ has only the first row as a member and $\mathcal{J}_{\mathrm{M}}(a_2)$ has only the second row as a member. So the two sets have an empty overlap. So $\mathcal{J}_{\mathrm{M}}(Ea_1a_2) = t$.)

We only need to associate matrices with normal chains whose first operator is basic. For every normal chain can be transformed into a normal chain which is satisfied by exactly the same models as the original chain. (If $\langle x_1, \ldots, x_n\rangle$ has a member x_i formed with a basic operator, form the chain $\langle x_i, \ldots, x_n, x_1, \ldots, x_{i-1}\rangle$; if there is no member of $\langle x_1, \ldots, x_n\rangle$ formed with a basic operator, then if $x_i = Q'ab$ let $y_i = Qba$ and form the chain $\langle y_n, y_{n-1} \ldots, y_1\rangle$.) There are two parts of the procedure for constructing the associated matrix

$$\begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \\ z_1 & z_2 & \ldots & z_n \end{vmatrix}$$

Part I, which makes assignments to every entry other than z_{n-1}, x_n, y_n, and z_n:

(i) Let $x_1 = b_1$ and $y_1 = \emptyset$.

(ii) For $1 \leq j < n-1$, let $z_j = \emptyset$.

(iii) If $Qb_{j-1}b_j$ is a sentence in the n-tuple, where Q is a positive operator and $1 < j \leq n-1$, then: a) If $x_{j-1} = b_{j-1}$, let $x_j = b_j$; otherwise, let $x_j = \emptyset$, and b) If $y_{j-1} = b_{j-1}$, let $y_j = b_j$; otherwise, let $y_j = \emptyset$.

(iv) If $Qb_{j-1}b_j$ is a sentence in the n-tuple, where Q is a negative operator and $1 < j \leq n-1$, then: a) If $x_{j-1} = b_{j-1}$, let $x_j = \emptyset$; otherwise, let $x_j = b_j$, and b) If $y_{j-1} = b_{j-1}$, let $y_j = \emptyset$; otherwise, let $y_j = b_j$.

Given this procedure we can see that $\mathcal{J}_{\mathrm{M}}$ assigns t to any sentence with operator Q_1 to Q_{n-2} provided $z_{n-1} = \emptyset$. But we will need to give special consideration to the portion of the second part of the procedure (below) that makes $z_{n-1} = b_{n-1}$.

The second part of the procedure makes assignments to z_{n-1}, x_n, y_n, and z_n. First, we list what operators can occur as Q_{n-1} and Q_n, ignoring E' and I' for the moment. If Q_1 is A, then Q_n is O (given the irreducibility of the chain), and so Q_{n-1} is A, E, I, O, or O'. So $Q_{n-1}Q_n$ may be AO, EO, IO, OO, or $O'O$. If Q_1 is E, then Q_n is E, O, or O'. So $Q_{n-1}Q_n$ may be EE, OE, $O'E$, EO', IO', OO', $A'O'$, or $O'O'$. If Q_1 is I, then $Q_n = A$, I, O, or O'. So $Q_{n-1}Q_n$ may be OA, AI, II, OI, or $O'I$. If Q_1 is O, then $Q_{n-1} = A$, E, I, O, or O'. No additional candidates for $Q_{n-1}Q_n$ appear under this condition. So the only candidates for $Q_{n-1}Q_n$ are those listed together with those formed by replacing I by I' or E by E'. Since any model which satisfies the stated pairs will satisfy the pairs formed by these replacements we only need to consider the pairs listed.

Call the two partially filled-in matrices generated by the first part of the procedure L and R:

$$\begin{array}{cc} L & R \\ \begin{vmatrix} b_1 & \dots & b_{n-1} & x_n \\ \emptyset & \dots & \emptyset & y_n \\ \emptyset & \dots & z_{n-1} & z_n \end{vmatrix} & \begin{vmatrix} b_1 & \dots & \emptyset & x_n \\ \emptyset & \dots & b_{n-1} & y_n \\ \emptyset & \dots & z_{n-1} & z_n \end{vmatrix} \end{array}$$

(L is generated iff the number of negative operators in $Q_1, \ldots, Q_{n-2}$ is even.) To fill in the remaining four gaps we make use of the following matrices, call them 1, 2, and 3:

$$\begin{vmatrix} b_n \\ b_n \\ \emptyset \end{vmatrix} \begin{vmatrix} \emptyset \\ \emptyset \\ b_n \end{vmatrix} \begin{vmatrix} b_n \\ \emptyset \\ \emptyset \end{vmatrix}$$

We put the candidates for $Q_{n-1}Q_n$ into four partitions and construct a three-rowed, n-columned matrix for the members of each partition. CONSTRUCTIONA 1: If $Q_{n-1}Q_n$ is AI, [II], AO, [IO], or $O'I$, let $z_{n-1} = \emptyset$, and either combine L and 1 or combine R and 1. (The brackets indicate that

II is a subordinate to AI and that IO is subordinate to AO. The work of checking the adequacy of the model can be simplified by recognizing the subordinates.)

So, for example, consider the irreducible chain $\langle Ea_1a_2, Oa_2a_3, Aa_3a_4, Oa_4a_1\rangle$. Part I of the procedure yields this matrix of type L:

$$\begin{vmatrix} a_1 & \emptyset & ax_3 \\ \emptyset & a_2 & \emptyset \\ \emptyset & \emptyset & \end{vmatrix}$$

Part II of the procedure, combining L and 1, yields:

$$M: \quad \begin{vmatrix} a_1 & \emptyset & a_3 & a_4 \\ \emptyset & a_2 & \emptyset & a_4 \\ \emptyset & \emptyset & \emptyset & \emptyset \end{vmatrix}$$

Refer to the rows as $R1$, $R2$, and $R3$. Then $\mathcal{J}_{\mathrm{M}}(a_1) = \{R1\}$, $\mathcal{J}_{\mathrm{M}}(a_2) = \{R2\}$, $\mathcal{J}_{\mathrm{M}}(a_3) = \{R1\}$, and $\mathcal{J}_{\mathrm{M}}(a_4) = \{R1, R2\}$. Note that $\mathcal{J}_{\mathrm{M}}$ assigns t to each sentence in the sequence.

$\mathcal{J}_{\mathrm{M}}$ meets condition i) for an interpretation: $\mathcal{J}_{\mathrm{M}}(b_i)$ is non-empty since b_i occurs in at least one of the rows of M and $\mathcal{J}_{\mathrm{M}}(b_i)$ is non-universal since b_i occurs in at most two rows of M. Given the procedure for constructing M, $\mathcal{J}_{\mathrm{M}}$ assigns t to all of the sentences in the original sequence. These remarks are also true of the interpretations given by the following three constructions.

CONSTRUCTION 2. If $Q_{n-1}Q_n$ is EE, $[EO, EO', OE, O'E, OO, OO', O'O$, or $O'O']$, let $z_{n-1} = \emptyset$, and either combine L and 2 or combine R and 2. CONSTRUCTION 3. If $Q_{n-1}Q_n$ is $A'O'$, or $[IO']$, let $z_{n-1} = b_{n-1}$, and either combine L and 2 or combine R and 2. (Note that $Q_{n-2} \neq A'$, for $A'A'$ reduces to A' and $A'I$ reduces to I.) CONSTRUCTION 4. If $Q_{n-1}Q_n$ is OA or $[OI]$, let $z_{n-1} = b_{n-1}$, and either combine L and 3 or combine R and 3. (Note that $Q_{n-2} \neq A'$, for $A'O$ reduces to O.) □

LEMMA 4. *If X is a normal chain, X reduces to Y, and Y is satisfiable in a model with a three-membered domain, then so is X.*

PROOF. The proof is by induction on the number of chains in a sequence that reduces X to Y.

BASIS STEP. Suppose X immediately reduces to Y. Then X has form $\ldots Q_1ab, Q_2bc\ldots$ and Y has form $\ldots Q_3ac\ldots$, where $Q_1Q_2Q_3 = AAA$, AE(or E')E, $AO'O'$, E(or E')AO', E(or E')I(or I')O', E(or E')$A'E$, I(or

I')AI, I(or I')E(or E')O, $OA'O$, $A'AI$, $A'E$(or E')O, $A'I$(or I')I, $A'OO$, $A'A'A'$, $O'AO'$. Assume there is a model $\langle D, \mathcal{J} \rangle$, where D has three members, such that $\mathcal{J}(Q_3ac) = t$. Then if Q_1 is positive, let $\mathcal{J}'(x) = \mathcal{J}(x)$ if $x \neq b$, and let $\mathcal{J}'(b) = \mathcal{J}(a)$. If Q_1 is negative and Q_2 is positive, let $\mathcal{J}'(x) = \mathcal{J}(x)$ if $x \neq b$, and let $\mathcal{J}'(b) = \mathcal{J}(c)$. Then if every sentence in Y is satisfied in $\langle D, \mathcal{J} \rangle$, every sentence in X is satisfied in $\langle D, \mathcal{J}' \rangle$.

INDUCTION STEP. Use the above reasoning. □

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