DISSERTATION

LAYERED BEAM VIBRATIONS INCLUDING SLIP

Submitted by

William Murray Henghold

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY WILLIAM MURRAY HENGHOLD ENTITLED LAYERED BEAM VIBRATIONS INCLUDING SLIP BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Committee on Graduate Work

Adviser

Head of Department

ABSTRACT

LAYERED BEAM VIBRATIONS INCLUDING SLIP

This study presents a theory for vibrations of layered beams with the effects of interlayer slip included. The theory is developed using Bernoulli-Euler assumptions with additional developments to account for the interlayer movement. The development is general, in the small deflection sense, for beams with mechanical connections.

The development leads to governing equations for beams having both dual and single axes of symmetry and an arbitrary number of layers. Solutions to the governing equations are presented in closed form for various sets of boundary conditions. These solutions show the effect of interlayer connection on the natural frequency, and it is shown that the solutions reduce to well known values for the extremes of interlayer connection.

The effect of damping on the solutions is presented, and the equations for the damped system are solved for small damping.

The results of some simple tests which were performed are presented, and these results are shown to agree favorably with the proposed theory.

William Murray Henghold
Civil Engineering Department
Colorado State University
Fort Collins, Colorado 80521
June, 1972

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

The use of layered systems for construction has become increasingly prevalent in modern structural engineering. Perhaps the best known examples of this are the aerospace industries' "sandwich" structures which combine high-strength facings with a light weight core. In civil engineering structures, the practice of building beams by nailing, bolting or gluing layers of wood together is used extensively.

The analysis of layered structures is nearly always based upon the assumption that perfect continuity exists at the interfaces of the connecting layers. If the layers are fastened together with strong adhesives, the rigid interconnection assumption is reasonable. This fact is borne out by such diverse works as Calcote (5),* Abel (7), Hoff and Mautner (15), Ross, Unger and Kerwin (16), and Raville, Ueng and Lei (17). For some widely used systems such as nailed wood construction, the assumption is subject to grave doubt. The interlayer movement has been shown to have a significant effect on the overall structural behavior (1, 2).

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*Numbers in parentheses correspond to references listed.

The objective of this study is to develop a general theory for the analysis of the vibrations of layered beams, including the effect of interlayer slip. The governing equations are developed and solved for various cases of interest. A numerical technique is presented to give approximate answers which include the effect of variation of beam properties along the beam length.

No extensive experimental work was attempted, rather a few simple experiments were conducted. These tests showed agreement with the proposed theory. The theory is general in the small deflection sense and is applicable to any layered beam with mechanical connections.

1.2 Literature Review

In this section a review will be made of major developments related to this study. Additional comments concerning some of these works will be made in a later section.

The statics of layered beam systems have been treated by several authors. These works were developed separately but have been shown to have striking similarities.

Granholm (22) developed a theory for layered systems including the effect of interlayer slip. His development was based upon the assumptions of doubly symmetric cross sections, a linear relationship between the force on a connector and its deformation, constant connector spacing and continuous shear connection between layers. The latter assumption implies a smoothing out of the discrete connection effects. These assumptions lead to the following governing equations for a system consisting of two equal layers:

$$\frac{d^2\varphi}{dx^2} - \frac{2bk}{EA}\varphi = r\frac{d^3Y}{dx^3} \qquad 1.1$$

$$\frac{d^2Y}{dx^2} - \frac{EAr}{2EI_s}\frac{d\varphi}{dx} = -\frac{M}{EI_s} \qquad 1.2$$

where

φ = relative longitudinal displacement between the layers (in),

b = the width of each layer (in),

k = displacement modulus (lb/in^2), related to φ by the relation $\tau = k\varphi$,

τ = shear flow between the layers (lb/in),

r = distance between the centroids of the two layers (in),

E = the modulus of elasticity of the material of the layers (lb/in^2),

I_s = moment of inertia of an equivalent solid section (in^4),

A = cross section area of each layer (in^2)

and

M = external moment at the section (in-lb).

Equations 1.1 and 1.2 can be solved simultaneously for the system deflection.

Pleshkov (23) also developed a theory for layered beams with interlayer slip. Again the assumptions of continuous shear connection, constant connector spacing and a linear connector force versus connector displacement relationship were assumed. The solution was generalized to include systems with only a vertical axis of symmetry. Generalization was also made to a system of n layers. Pleshkov's development led to the following governing equation:

$$E \sum_{k=1}^{n} i_k \frac{d^4Y}{dx^4} - \frac{4G}{E\Omega} \left(EI_s \frac{d^2Y}{dx^2} + M\right) = - \frac{d^2M}{dx^2} \qquad 1.3$$

where

E = the modulus of elasticity of the material (lb/in^2),

i_k = the moment of inertia of the kth layer about its own neutral axis (in^4),

I_s = the moment of inertia of the solid section (in^4),

M = the external moment (lb-in),

G = average connector modulus = $\dfrac{G_1 + G_2 + \dots G_n}{n}$,

G_k = connector modulus for the joint between the kth and k + 1st layers (lb/in^2),

T = connector shear flow (lb/in),

δ = connector displacement (in),

$\Omega = \dfrac{4}{n} \dfrac{A_1Z_1}{r_1} + \dfrac{A_1Z_1 + A_2Z_2}{r_2} + \dots + \dfrac{A_1Z_1 + \dots A_nZ_n}{r_n}$,

n = number of joints in the system (equal number of layers - 1),

A_k = area of the kth element (in^2),

r_k = the distance from centroid of the kth layer to that of the (k + 1)st layer

and

Z_k = the distance from the centroid of the kth element to the centroid of the entire section.

Newmark, Seiss and Viest (20, 21) have studied the problem of incomplete interaction between the steel girder and concrete slab of a composite T-beam. This problem is equivalent to a beam with interlayer slip in that the incomplete interaction between girder and slab is essentially the same as slip between beam layers. The authors verified their results with numerous small scale tests. Here again the assumptions of continuous shear connection and linear connector load versus connector deflection were assumed.

The most general treatment of the interlayer slip problem was made by J. R. Goodman (1, 2). For the case of systems of three equal layers he developed a comprehensive theory for beam, plate and shell systems. Experimental verification for wooden beam and plate systems was attempted with excellent results. He was able to treat non-linearities in the connector force versus connector deflection relationship by using a step wise linear numerical procedure. As an additional part of his study, he was able to show that the theories of Granholm (22), Pleshkov (23) and Newmark, Seiss and Viest (20) all provided the identical governing equation for a system of two equal layers. Again, this work used the continuous shear connection assumption. The close agreement between the author's theory and experiments showed that frictional effects were negligible for static bending.

For the case of a beam of three equal layers the following governing equation was developed:

$$3EI \frac{d^4y}{dx^4} - \frac{kn}{SEA} \left(EI_s \frac{d^2y}{dx^2} + M\right) = -\frac{d^2y}{dx^2} \qquad 1.4$$

where

E = the modulous of elasticity (lb/in^2),

I = the moment of inertia of an individual layer about its own neutral axis (in^4),

k = the connector modulous (lb/in),

n = the number of connectors per row,

S = the spacing between connector rows (in),

A = the area of an individual layer (in^2),

I_s = the moment of inertia of an equivalent solid beam (in^4)

and

M = the external moment at the section (in-lb).

Clark (18) has approached the interlayer slip problem from a different point of view. In his work, he assumed that slip took place between connectors but that the connectors themselves were perfectly rigid. This permitted the connectors to be considered discretely as opposed to using a continuous shear connection assumption. J. R. Goodman (1) showed, however, that the connector deformation was an important part of the slip problem for some systems and that Clark's work actually provided an upper bound for beam deflections.

Rassam (24) studied the problem of layered columns with interlayer slip. His study included columns with cross-sections having both single and double symmetry and allowed for variation in column properties along its length. Experimental verification of the developed theory was attempted and in general showed good agreement between experiment and theory. The study was limited to pinned-pinned columns but is applicable to any layered system. His development of the theory led to the following governing equation:

$$\frac{d^4y}{dx^4} + AP\frac{d^2y}{dx^2} - B\frac{d^2y}{dx^2} - CPY = 0 \qquad 1.5$$

where

y = the column deflection (in),

P = the column load (lb),

A = a constant ($1/in^2$-lb),

B = a constant ($1/in^2$)

and

C = a constant ($1/lb$-in^4).

The constants A, B and C depend upon the connection and properties of the layers and in general can become quite involved.

Another study, while not directly applicable to this effort, which is of general interest to those concerned with interlayer slip is the work of R. E. Goodman, Taylor and Brekke (25). This work is concerned with the development of a finite element model for jointed rock. The model allows for sliding joints and is concerned with finding failure conditions.

For the past twenty years or so extensive research effort has been expended to try to understand the mechanisms of damping in structural systems and to develop techniques to increase this dissipation of energy. The field of literature generated during these efforts is extremely large and diverse. Henderson (26) reports on the development of techniques in the areas of material damping, joint damping and the use of dissipative layers. He stresses that optimization of damping necessitates its inclusion in the earliest of design stages.

Material damping is more fully covered by Wood and Lee (13) and Lazan (3). Lazan hypothesizes that material damping is stress sensitive and may take the form

$$D = J\sigma^n \qquad 1.6$$

where

D = the specific damping ratio,

J = the damping constant,

σ = the stress level

and

n = the damping exponent.

It is noted that when $n = 2$ and D is not frequency dependent, equation 1.6 describes linear viscous damping.

The use of damping tapes and dissipative cores in sandwich structures is well reported as in the works of Plunkett (9) and Ross, Ungar and Kerwin (16). DiTaranto and Blasingame (6) report that the damping of laminated beams is essentially independent of boundary conditions.

The specific area of damping most relevant to this work is that dealing with joint or slip damping. Even so, these efforts tend to be highly specialized. L. E. Goodman (4) reviews various works in the area of slip damping. He points out that the energy loss is essentially due to Coulomb friction. This has the desirable effect of limiting vibration amplitude, coupled with the undesirable effects of fretting and corrosion of the structural members. Mentel (8) goes so far as to discard the technique of slip damping because of the supposed fretting problem.

Piann and Hallowell (29, 30) have investigated structural damping in simple built-up beams. The beam used was a cantilever with thin reinforcing spar caps. A heavy weight was attached to the end of the beam such that it could be treated as a single degree of freedom system. In the former paper it was assumed that sufficient clearance existed between the screws holding the beam together and their holes so as not to interfere with slip, while in the latter slip was hindered by the close fit of the screws. It was found that the damping was approximately proportional to the third power of the stress amplitude and inversely proportional to screw tightness at small amplitudes. At larger amplitudes the energy loss became proportional to the stress amplitude and the effect of screw tightness was effectively lost. It should be emphasized that the amplitudes in question were quite small such that slip was measured microscopically.

L. E. Goodman and Klumpp (27, 28) have studied slip damping as related to turbine blade vibration. The system used was a bileaf cantilever, clamped together with a number of clamps. This effectively simulated the turbine blade root and hub joint. Various clamping pressures were used. It was assumed that limiting shear stress and associated slip occurred only for a sufficiently large load. The amount of damping was found to depend on the normal pressure for any given load. At low pressures little damping was realized due to the small frictional force while at high pressure damping was diminished due to small slip. The increase in damping was significant in comparison to material damping and the results of this effort were helpful in reducing turbine blade failures by indicating the presence of an optimum pressure. The vibrational amplitudes here were again quite small with tip deflections on the order of .08 inches while slip was measured with a micrometer microscope.

Yen, Hartz and Brown (14) have studied the damping of wood T-beams. Again, interest is centered in very small but discernible amplitudes and "Goodman-Klumpp" type slip is assumed. The investigation includes such effects as nail spacing and nail bearing. The authors realized that the damping mechanism of their study was different from a linear viscous model but due to the range of damping ratios encountered and practicality of problem solution, they assumed an equivalent viscous model. They found that the bi-linear hysterisis loop reached an essentially reproducible state after a few cycles and that an elliptical representation of this loop was reasonable. The authors found that the damping ratios obtained were between .005 and .05 with the exact value being a function of nail bearing, nail spacing and amplitude.

Although the damping studies previously mentioned have each added to the understanding of damping in structures, this understanding is much more qualitative than quantitative. The predictability of damping from scale testing is still an elusive goal in many respects. This is especially true of more complex structures as seen from the fact that the Saturn V Apollo vehicle was dynamically tested in full scale (36).

CHAPTER 2

PRELIMINARIES AND THE SPECIAL CASE OF THREE EQUAL LAYERS

2.1 Introduction

In this chapter various assumptions and simplifications are introduced and discussed. Terms and concepts that form the basis of this study are defined and explained. The special case of a beam consisting of three equal layers is covered in detail. This detail is warranted by the fact that this system is easy to visualize, but at the same time it provides practically all the concepts necessary for a more general treatment. It is shown that the three equal layer system may be solved in closed form and solutions for four different sets of boundary conditions are presented. Finally, the boundary conditions are stated in an alternate form and the equivalence of results is shown.

2.2 Assumptions

In order to keep the problem within reasonable bounds, certain simplifying assumptions are made. These assumptions limit the analysis to a first order approximation. For the case of three equal layers the assumptions are:

1. The materials are linearly elastic and stresses remain within the elastic range.
2. The deflections are small.
3. The effects of shear deformation and rotary inertia are neglected.
4. The strain distribution through the depth of each layer is linear.
5. No separation between the layers is allowed.
6. The slip permitted by a connector is directly proportional to the load transmitted by the connector.
7. The connectors are equally spaced and of equal strength along the beam length.
8. A continuous, equivalent shear connection between the layers is assumed to replace the discrete connectors.
9. Friction effects are neglected.
10. The material and area properties are the same for each layer and are constant along the beam length.

Certain of the assumptions made deserve amplification so that the exact limitations of the proposed theory will be clear.

Neglecting the rotary inertia and transverse shear implies standard Bernoulli-Euler theory. This in turn implies that the beams in question are long in proportion to their depths. Timoshenko (10) found that the error in neglecting the rotary inertia and shear terms is on the order of 2% at the lower frequencies of vibration.

The assumption that no separation between the layers is allowed implies that each layer deflects the same amount. Additionally, when coupled with the small deflection limitation, it implies that each layer has the same curvature. The mechanical connectors between the layers generally prevent separation due to their resistance to withdrawal.

The assumption of a continuous shear connection means that the discrete deformable connections are replaced with a continuous shear connection. This is done for mathematical convenience. However, previous work, as explained in the literature review, has shown that this assumption is justified.

The neglecting of frictional effects definitely limits the class of problems to which the proposed theory is applicable. Systems involving high contact pressures or highly bi-linear hysterisis loops are not within the scope of this theory. However, a large class of problems still remain. J. R. Goodman's (1, 2) efforts with nailed wood structures indicate that the effect of friction between the layers is small. Additionally, bi-linear hysterisis loops may sometimes be replaced by elliptical ones and quite good results are still obtained.

The requirements of constant connector and beam properties along the length of the beam allow for a solution of the problem in closed form. The removal of these requirements necessitates a numerical solution to the problem. In a later section (3.5) the variation of the geometric and physical properties along the beam is considered.

2.3 Solid Beam

Perhaps the easiest way to see the concepts involved in the interlayer slip problem is to develop the governing equation for the free vibrations of a solid beam in a different manner than is normally encountered. Figure 2.1 shows a solid beam of a width b and a height 2h. A plane is inserted as shown such that the beam is divided into two equal beams of width b and height h.

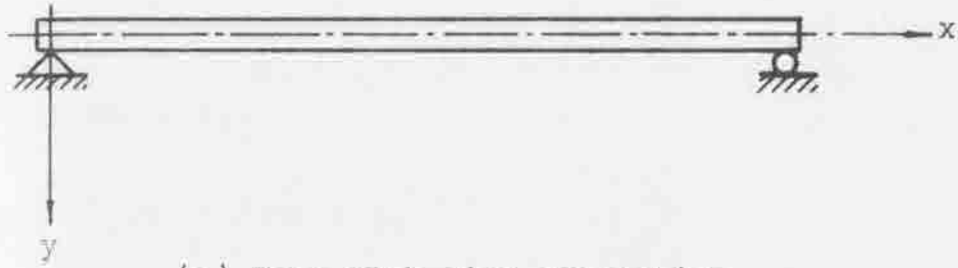
In Figure 2.1 some notation is introduced for the first time. This notation involves dependent variables subscripted with ,x. This subscripting indicates partial differentiation with respect to the indicated independent variable. The order of the differentiation is indicated by the number which preceeds the independent variable.

From horizontal equilibrium and the fact that the beam is solid, such that to first order the curvature of the upper and lower halves are the same, it is evident that

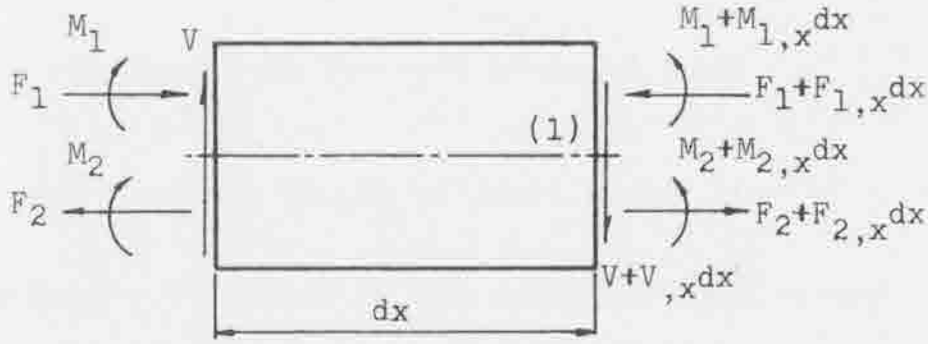
$$F_1 = F_2 = F$$

and

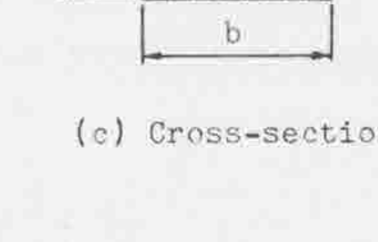
$$M_1 = M_2 = M.$$



(a) Beam with sign convention



(b) Beam element



(c) Cross-section

FIGURE 2.1 SOLID BEAM

Newton's second law is now applied as follows

$$\Sigma F_Y = ma_Y$$

$$V_{,x}dx = (mdx)Y_{,2t}$$

or $$V_{,x} = mY_{,2t} \qquad 2.1$$

where

Y = displacement of centroid of solid beam

and

m = mass per unit length of solid beam.

Now neglecting rotary inertia and shear deformation and summing moments gives

$$\Sigma M_{(1)} = 0$$

$$Vdx - 2M_{,x}dx - hF_{,x}dx = 0$$

$$V = 2M_{,x} + hF_{,x}. \qquad 2.2$$

It is now necessary to remove the dependent variable F from the problem. This is done by examining the strain at point 1 in figure 2.1 (b). Taking tension as positive, the strain is evaluated using the top and bottom halves of the beam as

$$\epsilon_1 = \frac{M}{EI}\frac{h}{2} - \frac{F}{EA} \qquad 2.3$$

$$\epsilon_1 = -\frac{M}{EI}\frac{h}{2} - \frac{F}{EA} \qquad 2.4$$

where

I = the moment of inertia of the half beam about its own centroid

$$= \frac{Ah^2}{12}$$

and

A = the half area

$= bh.$

Since the beam is solid, the strains expressed by equations 2.3 and 2.4 must be the same. This implies

$$F = \frac{Ah}{2I} M. \qquad 2.5$$

Equation 2.5 is now substituted into equation 2.2 to give

$$V = (2 + \frac{Ah^2}{2I}) M_{,x}. \qquad 2.6$$

Now for small deflections and the sign convention specified, the relationship

$$M = -EIY_{,2x}$$

is valid. Substitution of the above relationship into equation 2.6 and this in turn into equation 2.1 gives

$$(2 + \frac{Ah^2}{2I}) EIY_{,4x} + mY_{,2t} = 0. \qquad 2.7$$

It is noted that

$$(2 + \frac{Ah^2}{2I})\ I = 8I$$

$$= I_s$$

where

I_s = the moment of inertia of the solid beam about its centroid.

Thus equation 2.7 becomes

$$EI_sY_{,4x} + mY_{,2t} = 0. \qquad 2.8$$

Equation 2.8 represents the governing equation for the transverse vibrations of a solid beam with constant material and geometric properties along its length.

The equations for layered systems with interlayer slip will be derived in a similar manner with one outstanding exception. For a system with slip it is not possible to say that the strain at the joint between two layers is the same whether approached from the top or the bottom. A new relationship must be found.

2.4 The Three Equal Layer System

The governing equations will now be developed for the layered beam system consisting of three equal layers using the assumptions innumerated in Section 2.2. Figure 2.2 depicts the layered system and the associated beam forces

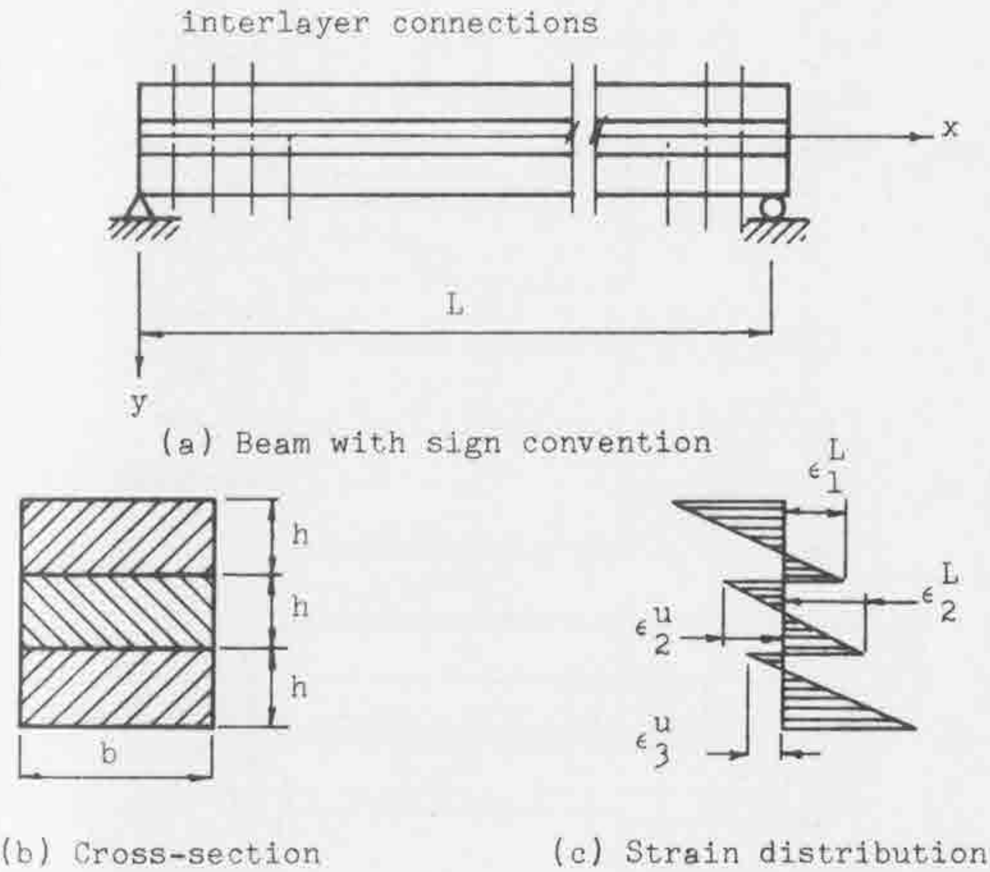
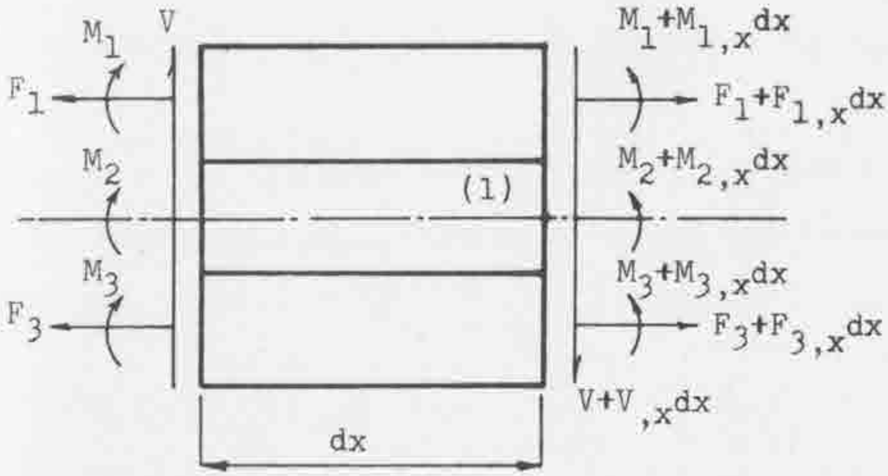
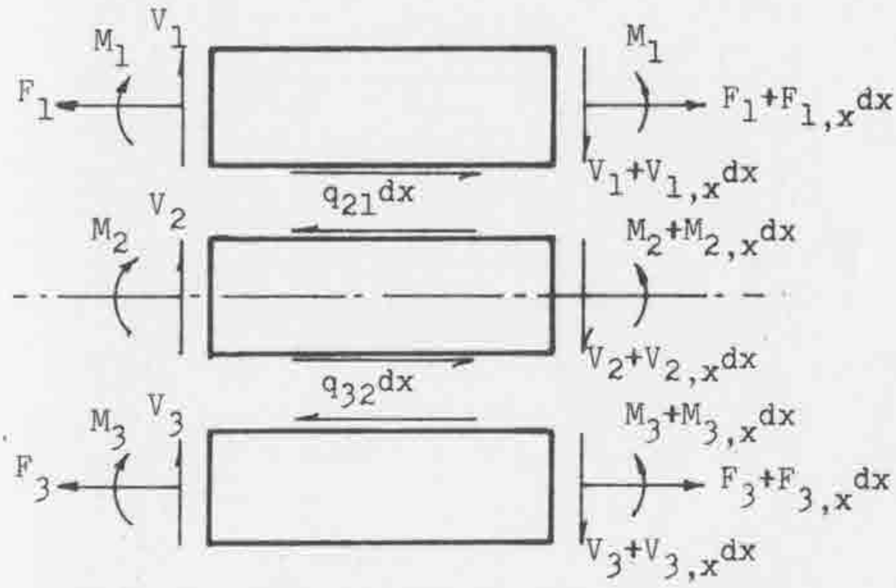


FIGURE 2.2 THREE EQUAL LAYER SYSTEM



(d) Beam element



(e) Layer elements

FIGURE 2.2--Continued

and strain distribution. It is noted that no axial force exists at the centroid of the second layer. That this is indeed the case can be seen using the symmetry of the problem. Rotation about the z axis indicates that F_2 must be zero.

Consider first the free body diagram of a beam element as shown in Figure 2.2 (d). Newton's second law is now applied.

$$\Sigma F_x = 0$$

gives

$$F_1 + F_{1,x}dx + F_3 + F_{3,x}dx - F_1 - F_3 = 0$$

or

$$F_{1,x} = - F_{3,x}. \qquad 2.9$$

$$\Sigma F_y = ma_y$$

gives

$$V + V_{,x}dx - V = \sum_1^3 (\rho_i A_i dx)\, Y_{i,2t}$$

or

$$V_{,x} = \sum_1^3 \rho_i A_i Y_{i,2t} \qquad 2.10$$

where

ρ_i = mass per unit volume of the ith layer (slugs/in^3)

and

A_i = area of the ith layer (in^2).

For the case where rotary inertia and shear deformation effects are neglected (assumption 3)

$$\Sigma M_{(1)} = 0$$

implies

$$V dx + \frac{5h}{2} F_{1,x} dx + \frac{h}{2} F_{3,x} dx - \sum_{1}^{3} M_{i,x} dx = 0. \qquad 2.11$$

Substitution of equation 2.9 into equation 2.11 gives

$$V = 2hF_{3,x} + \sum_{1}^{3} M_{i,x} \qquad 2.12$$

where

h = layer height (in).

Each layer has the same properties and is assumed to deflect the same amount (assumption 5). The position of the centroid of any layer will differ by a constant from that of the system centroid at any time t and position x. Additionally, the deflections are small (assumption 2). The following relationships then hold:

$$\sum_{1}^{3} \rho_i A_i = 3\rho A$$

$$Y_{i,2t} = Y_{,2t}$$

$$M_i = -EIY_{,2x}$$

where

Y = the position of the system centroid (in),

I = the moment of inertia of an individual layer about its own centroid (in^4)

and

E = modulus of elasticity (lb/in^2).

With these relationships and the fact that the layer properties remain constant along the beam's length, equation 2.12 may be rewritten as

$$V = 2hF_{3,x} - 3EIY_{,3x}.$$

Finally, substitution into equation 2.10 gives

$$3EIY_{,4x} - 2hF_{,2x} + 3\rho AY_{,2t} = 0 \qquad 2.13$$

where F_3 has been denoted simply as F.

Equation 2.13 contains two dependent variables, Y and F. A relationship between these variables is needed. This relationship may be obtained from an investigation of the effect of the non-rigidity of the connectors which allows slip between the layers.

The connectors are assumed to have equal moduli and be equally spaced along the length of the beam (assumption 7). Then

$$\frac{k}{S} = \text{constant}$$

where

k = the connector modulus (lb/in)

and

S = the connector spacing (in).

It is assumed that the slip permitted by a connector is directly proportional to the connector load (assumption 6). Then

$$\Delta_S = Q/k \qquad 2.14$$

where

Δ_S = the interlayer slip (in)

and

Q = the connector load (lb).

If it is now assumed that a continuous shear connection exists between two adjacent layers (assumption 8), the following relationship may be written

$$q_{ij}S = Qn \qquad 2.15$$

where

q_{ij} = the force transmitted between the ith and jth layers of the beam per unit length of the beam (lb/in) and is comparable to shear flow

and

n = the number of connectors per row.

Equations 2.14 and 2.15 are now combined to give

$$\Delta S_{ij} = \left(\frac{S}{kn}\right)_{ij} q_{ij}. \qquad 2.16$$

Equation 2.16 is general and allows for variation of S, k or n between layers.

Now consideration of the horizontal equilibrium of the layer elements shown in Figure 2.2 (e) gives

$$-q_{32}dx - F_3 + F_3 + F_{3,x}dx = 0$$

or

$$q_{32} = F_{3,x}. \qquad 2.17$$

Similarly

$$q_{21} = -F_{1,x}. \qquad 2.18$$

Substitution of these relationships into equation 2.16 and noting that the connector properties do not change on a per layer basis yields

$$\Delta S_{21} = -\frac{S}{kn} F_{1,x}$$

and

$$\Delta S_{32} = \frac{S}{kn} F_{3,x}. \qquad 2.19$$

An additional relationship expressing the interlayer slip may be found by investigating the strains at the interface of two layers. The strains denoted in Figure 2.2 (c) may be written, for tension taken as positive, as

$$\epsilon_1^L = \frac{M_1}{EI}\frac{h}{2} + \frac{F_1}{EA} \quad ,$$

$$\epsilon_2^u = -\frac{M_2}{EI}\frac{h}{2} \quad ,$$

$$\epsilon_2^L = \frac{M_2}{EI}\frac{h}{2} \quad ,$$

and

$$\epsilon_3^u = -\frac{M_3}{EI}\frac{h}{2} + \frac{F_3}{EA} \quad ,$$

where

ϵ_i^L = the strain at the lower edge of the ith layer

and

ϵ_i^u = the strain at the upper edge of the ith layer.

An alternate definition for relative slip between two layers may be stated as

$$\Delta S_{ij} = \int_o^x \epsilon_i^u dx - \int_o^x \epsilon_j^L dx \ . \qquad 2.20$$

For the case where i = 3 and j = 2, equations 2.19 and 2.20 are equated to give

$$\frac{S}{kn} F_{3,x} = \int_o^x \epsilon_3^u dx - \int_o^x \epsilon_2^L dx \ . \qquad 2.21$$

Instead of using equation 2.21 in integro-differential form, the derivative of both sides will be taken. Performing this operation and making the proper substitutions yields

$$\frac{S}{kn} F_{3,2x} = -\frac{h}{2EI}(M_3 + M_2) + \frac{F_3}{EA} \ . \qquad 2.22$$

Similarly

$$-\frac{S}{kn} F_{1,2x} = -\frac{h}{2EI}(M_2 + M_1) - \frac{F_1}{EA} \ . \qquad 2.23$$

It is noted that

$$M_i = - EIY_{,2x}$$

and from equation 2.9

$$F_1 = - F_3$$

If these substitutions are made into equation 2.22 or 2.23, both equations reduce to the same form. That is

$$\frac{S}{kn} F_{,2x} = hY_{,2x} + \frac{F}{EA} \qquad 2.24$$

where the subscript has been dropped from F as unnecessary.

The fact that equations 2.22 and 2.23 reduced to the same equation is worthy of note. This indicates that the relative slip between the first and second layers is the same as that between the second and third. This is as it should be due to the symmetry of the system.

Equations 2.13 and 2.24 represent two equations in two unknowns Y and F and are the governing equations for the free vibrations of a three equal layer beam with interlayer slip.

2.5 Boundary Conditions

To obtain a complete solution to the equations developed, boundary conditions are needed. The boundary conditions for the system of three equal layers will of necessity be different from those of a simple Euler beam. This fact can readily be seen from an inspection of equations 2.13 and 2.24. To obtain a solution to these equations, a total of six boundary conditions are needed where only four are required for an Euler beam problem.

Consider first the conditons associated with a simply supported end. It is natural that the deflection and total moment on the end section are zero. From Figure 2.2(d) and horizontal equilibrium

$$M_T = 3M + 2hF$$

$$= 2hF - 3EIY_{,2x}.$$

The conditions at the end are then

$$Y = 0$$

and

$$3\ EIY_{,2x} - 2hF = 0.$$

One more condition is needed. This condition can be found by noting that at a simply supported end, there is nothing to give rise to or to support any axial force. Therefore, the axial forces must go to zero at the end. For F equal to zero the boundary conditions can then be written as

$$Y = 0\ ,$$

$$Y_{,2x} = 0$$

and

$$F = 0. \qquad 2.25$$

The conditions at a free end are investigated next. Again, there can be no axial force at the end and the vanishing of total moment and total shearing force is reasonable from experience with Euler type problems. For

$$V_T = 2hF_{,x} - 3EIY_{,3x}$$

the conditions then become

$$Y_{,2x} = 0 \ ,$$

$$3EIY_{,3x} - 2hF_{,x} = 0$$

and

$$F = 0 \ . \qquad 2.26$$

It is noted that the second of the above conditions is mixed in the sense that both dependent variables appear at the same time.

Next, consider a fixed end. Following the above reasoning it is easily seen that the displacement and slope of the end should be zero. Here, however, the axial force cannot be zero. The end of the beam is fixed against slip displacement. From equation 2.16 this condition on slip displacement indicates that

$$q_{ij} = 0 \ .$$

Horizontal equilibrium and equation set 2.19 in turn imply

$$F_{,x} = 0 \ .$$

The boundary conditions may now be summarized as

$$Y = 0 \ ,$$

$$Y_{,x} = 0$$

and

$$F_{,x} = 0 \ . \qquad 2.27$$

From equations 2.25 through 2.27 it is evident that for any set of consistent conditions a set of six boundary conditions are available as required.

As a check on the boundary conditions, the equations of motion for the three equal layer system were derived using the extended Hamilton principle. This method has the advantage of providing the natural boundary conditions needed to supplement the geometric boundary conditions as well as the differential equations of motion. This development can be found in Appendix II. The results obtained are in complete agreement with the results of this section.

2.6 General Solution for the Three Equal Layer System

The governing equations for the free vibrations of the three equal layer system were developed in Section 2.4 and are repeated here for easy reference.

$$3EIY_{,4x} - 2hF_{,2x} + 3\rho AY_{,2t} = 0 \ .$$

$$hY_{,2x} - \frac{S}{kn} F_{,2x} + \frac{F}{EA} = 0 \ .$$

It will prove convenient to work in terms of beams of length unity. To this end a change of variable is made such that

$$z = \frac{x}{L}$$

where

L = the length of the beam

and

1 = the normalized length of the beam.

Under this change of variable

$$dz = \frac{dx}{L} \quad ,$$

$$Y_{,x} = \frac{1}{L}\, Y_{,z}$$

and

$$Y_{,nx} = \left(\frac{1}{L}\right)^{n} Y_{,nz} \; .$$

The governing equations may then be written as

$$\frac{3EI}{L^4}\, Y_{,4z} - \frac{2h}{L^2}\, F_{,2z} + 3\rho A Y_{,2t} = 0$$

and

$$\frac{h}{L^2}\, Y_{,2z} = \frac{S}{knL^2}\, F_{,2z} + \frac{1}{EA}\, F = 0 \; . \qquad 2.28$$

For the case where the beam and connector properties do not change along the length of the beam, the above equations form a set of linear, constant coefficient, partial differential equations. For this type of equation the time dependency is separable or, more to the point, simple harmonic motion may be assumed such that without loss of generality

$$Y = Y(z)\ \cos(\omega t + \varphi)$$

and

$$F = F(z)\ \cos(\omega t + \varphi) \qquad 2.29$$

where

ω = the natural frequency (rad./sec.)

and

φ = some phase angle which will depend upon the initial conditions of the problem (rad.).

Substitution of equation set 2.29 into the governing equations 2.28 after collecting terms yields

$$\left(\frac{3EI}{L^4} Y^{IV} - 3\rho A\omega^2 Y - \frac{2h}{L^2} F''\right)\cos(\omega t + \varphi) = 0$$

and

$$\left(\frac{h}{L^2} Y'' - \frac{S}{knL^2} F'' + \frac{1}{AE} F\right) \cos(\omega t + \varphi) = 0 \qquad 2.30$$

where a superscript roman numeral or prime on one of the dependent variables indicates ordinary differentiation with respect to the normalized space variable z.

Since the term cos $(\omega t + \varphi)$ cannot be zero for all time t, the terms in parentheses must be zero. This leads to a pair of ordinary differential equations which may be written in matrix operator form as

$$\begin{bmatrix} \frac{3EI}{L^4} D^4 - 3\rho A\omega^2 & -\frac{2h}{L^2} D^2 \\ \\ \frac{h}{L^2} D^2 & -\frac{S}{knL^2} D^2 + \frac{1}{AE} \end{bmatrix} \begin{Bmatrix} Y \\ \\ F \end{Bmatrix} = \begin{Bmatrix} 0 \\ \\ 0 \end{Bmatrix} \qquad 2.31$$

where

$$D^n = \frac{d^n}{dz^n}$$

The matrix equation 2.31 may now be solved for the characteristic equation for Y or F by expanding the determinant of the operator matrix. After some algebraic manipulation this yields

$$[D^6 - \frac{knL^2}{SEA}(1 + \frac{2Ah^2}{3I}) D^4 - \frac{\rho aL^4}{EI}\omega^2 D^2 + \frac{knL^2}{SEA}\frac{\rho aL^4}{EI}\omega^2] Y = 0. \qquad 2.32$$

It is noted that

$$Ah^2 = 12 I$$

and

$$\frac{2Ah^2}{3I} = 8 .$$

Let

$$\frac{knL^2}{SEA} = K$$

and

$$\frac{\rho aL^4}{EI} = N.$$

Equation 2.32 may now be written as

$$[D^6 - 9KD^4 - N\omega^2D^2 + KN\omega^2]\ \ Y = 0 \quad . \qquad 2.33$$

Equation 2.33 may now be solved in the standard form by assuming a solution of the form

$$Y = \sum_{1}^{6} \bar{a}_i e^{d_i z}$$

where

$\bar{a}_i$ = a constant

and

d_i = the root of a polynomial.

Once a solution of the form assumed is substituted into equation 2.33 and the indicated differentiations are carried out, a sixth order polynomial in d is obtained such that

$$d^6 - 9Kd^4 - N\omega^2d^2 + KN\omega^2 = 0. \qquad 2.34$$

It is noted that only 2n powers of d exist in equation 2.34. The order of the equation may then be reduced by making the substitution

$$p = d^2 \ .$$

Equation 2.34 then becomes

$$p^3 - 9Kp^2 - N\omega^2p + KN\omega^2 = 0 \ . \qquad 2.35$$

which is a cubic which may be solved with relative ease.

In solving a cubic, one quantity of interest is the discriminant. If the sign of the discriminant can be determined, the general nature of the roots can then be stated. Let equation 2.35 be written in standard cubic form as

$$p^3 + a_1 p^2 + a_2 p + a_3 = 0$$

where

$$a_1 = -\,9K,$$
$$a_2 = -\,N\omega^2$$

and

$$a_3 = KN\omega^2$$

The discriminant is defined as

$$\text{Dis} = (q^*)^3 + (r^*)^2 \ . \qquad 2.36$$

where

$$q^* = \frac{3a_2 - a_1^2}{9} = -\left(\frac{N\omega^2}{3} + 9K^2\right)$$

and

$$r^* = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54} = KN\omega^2 + 27K^3.$$

Upon carrying out the operations indicated by equation 2.36 and after some algebraic manipulations, the following relationship is obtained.

$$\text{Dis} = -\,\frac{N\omega^2}{27}\,(N\omega^2 + 27K^2)^2 \ . \qquad 2.37$$

Now ω, K and N are real numbers and N is greater than zero. Thus, an inspection of equation 2.37 reveals that the discriminant for the cubic equation 2.35 is always less than zero which implies that the roots to equation 2.35 are all real and distinct. Denote these roots as

$$p_1 = \gamma^2 \ ,$$

$$p_2 = \alpha^2$$

and

$$p_3 = \beta^2 \ .$$

By a simple matter of expansion, it is easily shown that the following relationships between the roots hold.

$$p_1 + p_2 + p_3 = 9K$$

$$p_1 p_2 + p_2 p_3 + p_1 p_3 = - K\omega^2$$

$$p_1 p_2 p_3 = - KN\omega^2 \ . \qquad 2.38$$

The roots p_i have been shown to be real and distinct. Additionally, from their definitions K, N and ω^2 must be greater than zero. For these conditions, it is obvious from the first of equation set 2.38 that at least one root must be greater than zero. The last of the equations of set 2.38 requires either one root less than zero or all roots less than zero. The two conditions together can be satisfied only for one root less than zero and two roots greater than zero. Let these roots be

$$\gamma^2 < 0 \ ,$$

$$\alpha^2 > 0$$

and

$$\beta^2 > 0 \ .$$

Since p is equal to d^2, the roots to the sixth order polynomial given by equation 2.34 are

$$\pm\, i\gamma \ ,$$

$$\pm\, \alpha$$

and

$$\pm\, \beta \ .$$

and the solution for Y is then

$$Y = \bar{a}_1 e^{i\gamma z} + \bar{a}_2 e^{-i\gamma z} + \bar{a}_3 e^{\alpha z} + \bar{a}_4 e^{-\alpha z} + \bar{a}_5 e^{\beta z} + \bar{a}_6 e^{-\beta z} \ .$$

By a simple redesignation of the constants $\bar{a}_i$, the solution for Y may be rewritten as

$$Y = a_1 \sin(\gamma z) + a_2 \cos(\gamma z) + a_3 \sinh(\alpha z) + a_4 \cosh(\alpha z) + a_5 \sinh(\beta z) + a_6 \cosh(\beta z). \qquad 2.39$$

It is evident that the characteristic equation for F is the same as that for Y. Proceeding as before the functional form of F may be found as

$$F = b_1 \sin(\gamma z) + b_2 \cos(\gamma z) + b_3 \sinh(\alpha z) + b_4 \cosh(\alpha z) + b_5 \sinh(\beta z) + b_6 \cosh(\beta z). \qquad 2.40$$

Equations 2.39 and 2.40 contain a total of twelve arbitrary constants. From an inspection of matrix equation 2.31, it is clear that there are in general only six arbitrary constants in the solution of the three equal layer system. Therefore, some relationship must exist between the a_i's and the b_i's. This relationship may be found by substitution of equations 2.39 and 2.40 into one of the two equations indicated by 2.30. Substitution in the second equation yields

$$\frac{h}{L^2}\,(-\gamma^2 a_1\sin(\gamma z) - \gamma^2 a_2\cos(\gamma z) + \alpha^2 a_3\sinh(\alpha z) + \alpha^2 a_4\cosh(\alpha z)$$
$$+ \beta^2 a_5\sinh(\beta z) + \beta^2 a_6\cosh(\beta z)) - \frac{S}{knL^2}\,(-\gamma^2 b_1\sin(\gamma z)$$
$$- \gamma^2 b_2\cos(\gamma z) + \alpha^2 b_3\sinh(\alpha z) + \alpha^2 b_4\cosh(\alpha z)$$
$$+ \beta^2 b_5\sinh(\beta z) + \beta^2 b_6\cosh(\beta z)) + \frac{1}{AE}\,(b_1\sin(\gamma z)$$
$$+ b_2\cos(\gamma z) + b_3\sinh(\alpha z) + b_4\cosh(\alpha z) + b_5(\sinh(\beta z)$$
$$+ b_6\cosh(\beta z)) = 0.$$

Now collecting like terms gives

$$-\frac{h}{L^2}\,\gamma^2 a_1 + \left(\frac{S}{knL^2}\,\gamma^2 + \frac{1}{AE}\right) b_1 = 0$$

or

$$b_1 = \frac{AEh\gamma^2}{\dfrac{\gamma^2}{K} + 1}\, a_1$$

$$= c_1 a_1\,. \qquad 2.41$$

Similarly

$$b_2 = c_1 a_2 \ ,$$
$$b_3 = c_2 a_3 \ ,$$
$$b_4 = c_2 a_4 \ ,$$
$$b_5 = c_3 a_5$$

and

$$b_6 = c_3 a_6 \ .$$

where

$$C_2 = \frac{AEh\alpha^2}{\frac{\alpha^2}{K} - 1} \ , \qquad 2.42$$

and

$$C_3 = \frac{AEh\beta^2}{\frac{\beta^2}{K} - 1} \ . \qquad 2.43$$

In summary, the solution to the governing equations for the free vibrations of the three equal layer system is

$$Y = Y(z)\cos(\omega t + \varphi)$$

$$F = F(z)\cos(\omega t + \varphi)$$

where

$$Y = a_1 \sin(\gamma z) + a_2 \cos(\gamma z) + a_3 \sinh(\alpha z) + a_4 \cosh(\alpha z)$$
$$+ a_5 \sinh(\beta z) + a_6 \cosh(\beta z)$$

and

$$F = c_1(a_1 \sin(\gamma z) + a_2 \cos(\gamma z)) + c_2(a_3 \sinh(\alpha z)$$
$$+ a_4 \cosh(\alpha z)) + c_3(a_5 \sinh(\beta z) + a_6 \cosh(\beta z)) \qquad 2.44$$

with γ, α and β, the roots to the sixth order polynomial 2.34 and c_i as defined by equations 2.41 through 2.43.

The natural frequencies and associated modes of vibration are clearly dependent upon the boundary conditions of the problem. In Section 2.5 it was shown that the six boundary conditions necessary for a complete solution are available. Various sets of boundary conditions can now be investigated.

Consider first a simply supported beam. For a beam simply supported at both ends, the boundary conditions which must be met are given by 2.35 as

$$Y = 0 \ ,$$
$$Y'' = 0$$

and

$$F = 0$$

at $\qquad z = 0$ and $z = 1$.

Substitution of equation set 2.44 into these six conditions and writing the results in matrix form yields

$$[BC] \left\{a\right\} = 0 \qquad 2.45$$

where

$$a = \left\{\begin{matrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_6 \end{matrix}\right\}$$

and

$$[BC] = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -\gamma^2 & 0 & \alpha^2 & 0 & \beta^2 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ \sin(\gamma) & \cos(\gamma) & \sinh(\alpha) & \cosh(\alpha) & \sinh(\beta) & \cosh(\beta) \\ -\gamma^2\sin(\gamma) & -\gamma^2\cos(\gamma) & \alpha^2\sinh(\alpha) & \alpha^2\cosh(\alpha) & \beta^2\mathrm{sihn}(\beta) & \beta^2\cosh(\beta) \\ c_1\sin(\gamma) & c_1\cos(\gamma) & c_2\sinh(\alpha) & c_2\cosh(\alpha) & c_3\sinh(\beta) & c_3\cosh(\beta) \end{bmatrix} .$$

In order that equation 2.45 have a non-trivial solution, the determinant of BC must be zero, i.e.

$$\left| BC \right| = 0 . \qquad 2.46$$

The solution of equation 2.46 yields, after some algebraic simplifications

$$[(\frac{\alpha^2}{\gamma^2} + 1)\ (\frac{c_3}{c_1} - 1) - (\frac{\beta^2}{\gamma^2} + 1)\ (\frac{c_2}{c_1} - 1)]$$

$$\sin(\gamma)\sinh(\alpha)\sinh(\beta) = 0 . \qquad 2.47$$

The first term in equation 2.47 will be zero, only for the case when α^2 is equal to β^2. Since α^2 and β^2 are distinct, the bracketed term is not zero. This leaves

$$\sin(\gamma)\sinh(\alpha)\sinh(\beta) = 0 . \qquad 2.48$$

The term $\sinh(\alpha)$ will be zero only for the case when $\alpha = 0$. An investigation of equation 2.34 shows that a zero root will be obtained only for ω^2 equal to zero since K and N are not zero. The solution for $\omega^2 = 0$ is of trivial importance for the case of a simply supported beam. Therefore

$$\sinh(\alpha) \neq 0 .$$

Similarly

$$\sinh(\beta) \neq 0 .$$

This then leaves

$$\sin(\gamma) = 0 \ . \qquad 2.49$$

Equation 2.49 has an infinite number of solutions. These solutions are

$$\gamma = n\pi$$

where

$$n = 1,2,3 \ \ldots \qquad 2.50$$

It is now a simple matter to back substitute into equation 2.45 to determine the mode shapes. Carrying out this operation yields

$$Y = a_1 \sin(n\pi z)$$

$$F = c_1 a_1 \sin(n\pi z) \ . \qquad 2.51$$

Of prime interest to the problem are those values of ω^2 (the eigenvalues) which provide solutions to the problem. Solution of equation 2.35 for ω^2 yields

$$\omega^2 = \frac{p^2}{N} \left(\frac{p - 9K}{p - K}\right) \ .$$

Now the root of interest is given by equation 2.50. Since p is equal to d^2, the substitution $-(n\pi)^2$ is made for p. This gives

$$\omega^2 = \frac{(n\pi)^4}{N} \left(\frac{(n\pi)^2 + 9K}{(n\pi)^2 + K}\right) \ . \qquad 2.52$$

Equation 2.52 provides the eigenvalues associated with the simply supported end conditions. It is of interest to investigate some limiting cases of this equation. Suppose that the value of K gets extremely small. This corresponds to the interlayer connection approaching zero. Equation 2.52 then becomes

$$\omega^2 = \frac{(n\pi)^4}{N}$$

Substitution for N and solving for ω gives

$$\omega = (n\pi)^2 \sqrt{\frac{EI}{\rho A L^4}} \qquad 2.53$$

Equation 2.53 represents the natural frequency of vibration for a simply supported beam of width b and height h. This verifies that the solution limits properly at the lower limit of no connection between the layers.

Suppose instead that K grows very large representing the case of completely rigid interlayer connection. Equation 2.52 can then be approximated as

$$\omega^2 = \frac{9(n\pi)^4}{N} \ .$$

Substitution for N and solution for ω yields

$$\omega = (n\pi)^2 \sqrt{\frac{9EI}{\rho A L^4}} \ ,$$

$$= (n\pi)^2 \sqrt{\frac{27EI}{3\rho A L^4}} \ ,$$

or

$$\omega = (n\pi)^2 \sqrt{\frac{EI_s}{\rho A_s L^4}} \tag{2.54}$$

where

I_s = the moment of inertia of a beam of width b and height 3h

A_s = the area of a beam of width b and height 3h.

Equation 2.54 shows that the solution limits properly to the value of the natural frequency for a solid beam for the case of very rigid connection.

Given the constants which make up N and K, it is possible to calculate all the eigenvalues from equation 2.52. For convenience in plotting, the dimensionless factor Ψ will be defined as

$$\Psi = \frac{\omega^2}{\omega_L^2} \tag{2.55}$$

where

ω_L = the natural frequency of a solid beam equivalent to one of the individual layers of the system.

With the aid of the definition given by equation 2.55, substitution into equation 2.52 gives

$$\Psi = \frac{(n\pi)^2 + 9K}{(n\pi)^2 + K} \; . \tag{2.56}$$

A plot of Ψ versus K is shown in Figure 2.3 for the first two modes of the pinned-pinned beam. In general, the connectors become less effective at higher modes.

The eigenvalues and eigenvectors for alternate sets of boundary conditions are found in exactly the same manner as for the simply supported case. In each case the boundary condition equations are written in the form

$$[BC] \left\{a\right\} = \left\{0\right\} \quad .$$

The eigenvalues are then found by equating the determinant of the B. C. matrix to zero.

The boundary condition at a free end which involves both Y and F requires some algebraic juggling. By denoting I as $\frac{bh^3}{12}$, this condition may be written as

$$Y''' - \frac{8}{AEh} F' = 0. \qquad 2.57$$

Now using the values given by equation set 2.44, carrying out the indicated differentiation and substituting in equation 2.57 yields after some manipulation

$$e_1(-a_1\cos(\gamma z) + a_2\sin(\gamma z)) + e_2(a_3\cosh(\alpha z) + a_4\sinh(\alpha z))$$

$$+ e_3(a_5\cosh(\beta z) + a_6\sinh(\beta z)) = 0$$

where

$$e_1 = \gamma^3 \left(\frac{\gamma^2 + 9K}{\gamma^2 + \gamma}\right) ,$$

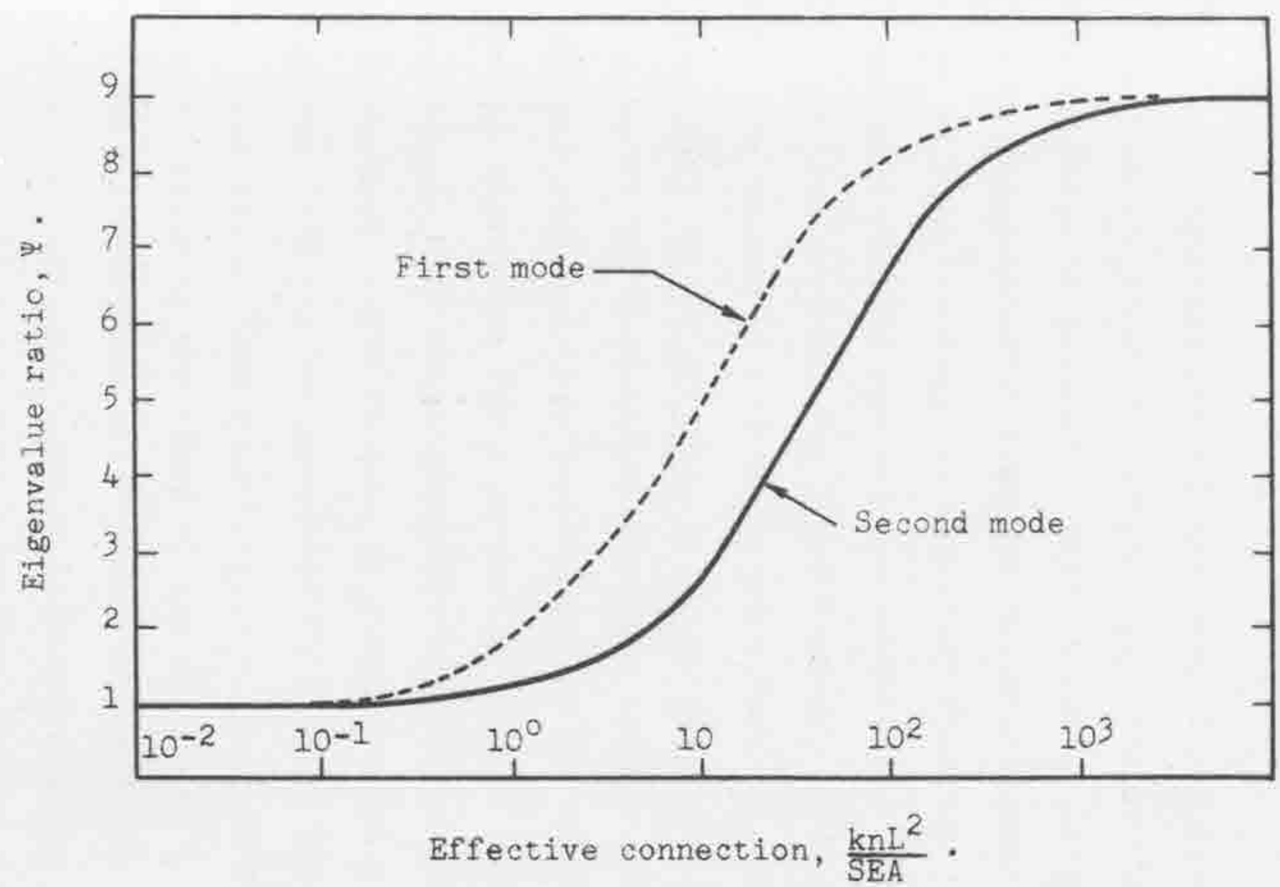


FIGURE 2.3 EIGENVALUE RATIO VERSUS EFFECTIVE CONNECTION FOR A SIMPLY SUPPORTED BEAM

$$e_2 = \alpha^3 \left(\frac{\alpha^2 - 9K}{\alpha^2 - K}\right)$$

and

$$e_3 = \beta^3 \left(\frac{\beta^2 - 9K}{\beta^2 - K}\right) .$$

The B. C. matrices for a fixed-free, fixed-fixed and free-free beam are shown in Figure 2.4. When the determinant of these matrices is equated to zero, a rather large and complicated transcendental equation results in each case. This type of equation requires numerical solution.

A computer program was written to solve for the eigenvalues. In this program, for K and N given, a value of ω was assumed. The values of α, β and γ were then calculated and the determinant of the specific B. C. matrix found. Output of the value of the determinant versus ω showed where the value changed sign and thus the eigenvalue was determined.

Once an eigenvalue for a particular set of boundary conditions is known, its associated mode shape or eigenvector can be determined. This is done by simply removing one of the equations from the set generated in forming the B. C. matrix. This leaves five equations in six unknowns which may be solved by assuming the value of one of the a_i's. The set of simultaneous equations which generate the mode shape of interest was solved using another computer program.

Plots of Ψ (as defined by equation 2.55) versus K were generated for fixed-free, fixed-fixed, and free-free end conditions from computer results. In each case, the value

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ \gamma & 0 & \alpha & 0 & \beta & 0 \\ c_1\gamma & 0 & c_2\alpha & 0 & c_3\beta & 0 \\ -\gamma^2\sin(\gamma) & -\gamma^2\cos(\gamma) & \alpha^2\sinh(\alpha) & \alpha^2\cosh(\alpha) & \beta^2\sinh(\beta) & \beta^2\cosh(\beta) \\ -e_1\cos(\gamma) & e_1\sin(\gamma) & e_2\cosh(\alpha) & e_2\sinh(\alpha) & e_3\cosh(\beta) & e_3\sinh(\beta) \\ c_1\sin(\gamma) & c_1\cos(\gamma) & c_2\sinh(\alpha) & c_2\cosh(\alpha) & c_3\sinh(\beta) & c_3\cosh(\beta) \end{bmatrix}$$

(a) Fixed-free beam

FIGURE 2.4 MATRICES FOR VARIOUS END CONDITIONS

$$
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
\gamma & 0 & \alpha & 0 & \beta & 0 \\
c_1\gamma & 0 & c_2\alpha & 0 & c_3\beta & 0 \\
\sin(\gamma) & \cos(\gamma) & \sinh(\alpha) & \cosh(\alpha) & \sinh(\beta) & \cosh(\beta) \\
\gamma\cos(\gamma) & -\gamma\sin(\gamma) & \alpha\cosh(\alpha) & \alpha\sinh(\alpha) & \beta\cosh(\beta) & \beta\sinh(\beta) \\
c_1\gamma\cos(\gamma) & -c_1\gamma\sin(\gamma) & c_2\alpha\cosh(\alpha) & c_2\alpha\sinh(\alpha) & c_3\beta\cosh(\beta) & c_3\beta\sinh(\beta)
\end{bmatrix}
$$

(b) Fixed-fixed beam

FIGURE 2.4--Continued

$$
\begin{bmatrix}
0 & -\gamma^2 & 0 & \alpha^2 & 0 & \beta^2 \\
-e_1 & 0 & e_2 & 0 & e_3 & 0 \\
0 & c_1 & 0 & c_2 & 0 & c_3 \\
-\gamma^2\sin(\gamma) & -\gamma^2\cos(\gamma) & \alpha^2\sinh(\alpha) & \alpha^2\cosh(\alpha) & \beta^2\sinh(\beta) & \beta^2\cosh(\beta) \\
-e_1\cos(\gamma) & e_1\sin(\gamma) & e_2\cosh(\alpha) & e_2\sinh(\alpha) & e_3\cosh(\beta) & e_3\sinh(\beta) \\
c_1\sin(\gamma) & c_1\cos(\gamma) & c_2\sinh(\alpha) & c_2\cosh(\alpha) & c_3\sinh(\beta) & c_3\cosh(\beta)
\end{bmatrix}
$$

(c) Free-free beam

FIGURE 2.4--Continued

of Ψ had limits of 1 and 9. These have been shown previously to be the proper limiting values for no connection and rigid connection respectively. The plots had essentially the same form as shown in Figure 2.3. Typical results are shown in Figure 2.5 for a fixed-fixed beam. Typical mode shapes generated from computer results are shown in Figure 2.6.

Copies of the computer programs used are shown in Appendix III.

2.7 Solution for Boundary Conditions in Alternate Form

It will prove advantageous in a later section to use the boundary conditions in a form other than those shown in equations 2.25 through 2.27. To this end the boundary conditions will be stated as functions of Y only, and the equivalence of these conditions will be shown by the fact that the solutions remain the same as those generated in the previous section.

If the governing equations are written in the form given by equation set 2.30 and F is solved for, the resulting equation is

$$F = \frac{3EI}{2hKL^2}\left(Y^{IV} - 8KY'' - N\omega^2 Y\right) \qquad 2.58$$

where K and N are as previously defined

For a pinned end the boundary conditions are given by equation 2.25 as

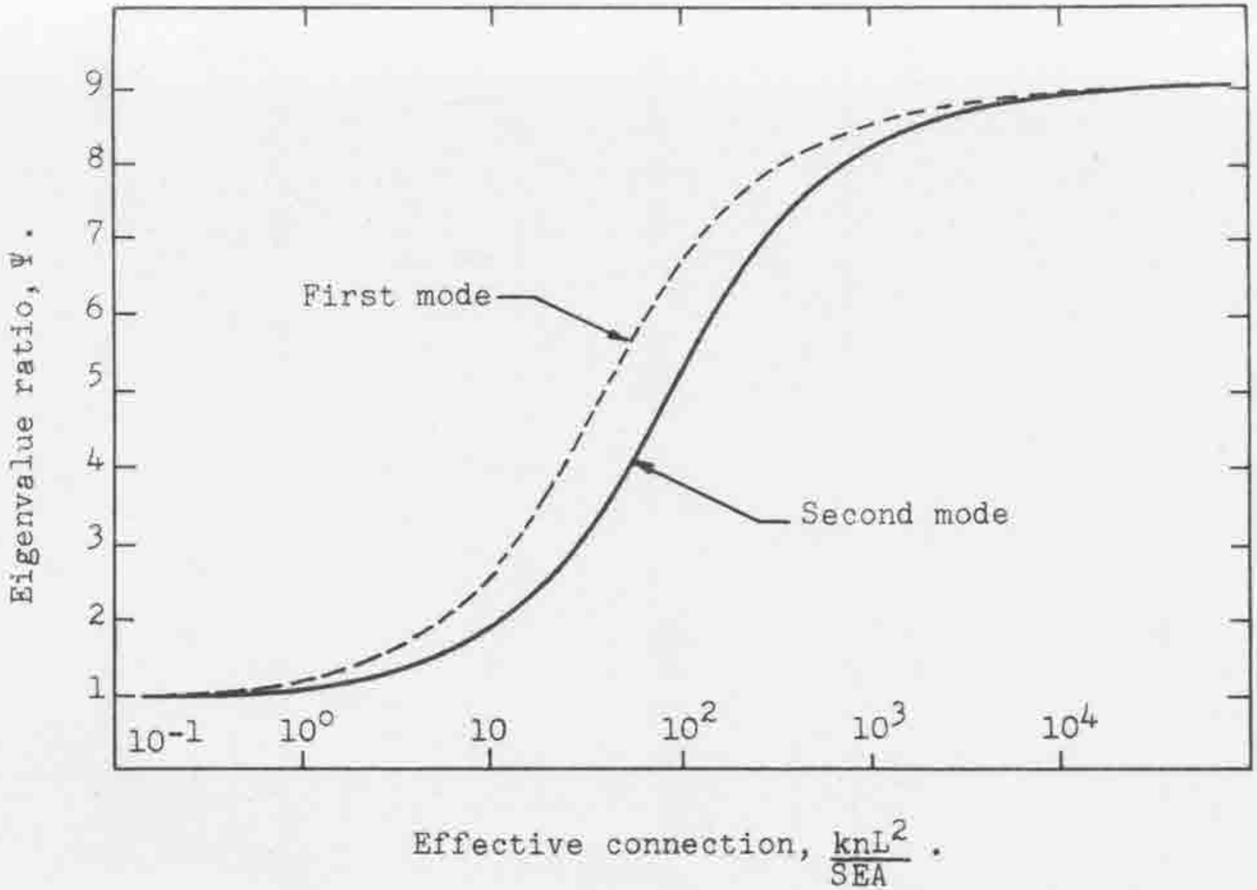
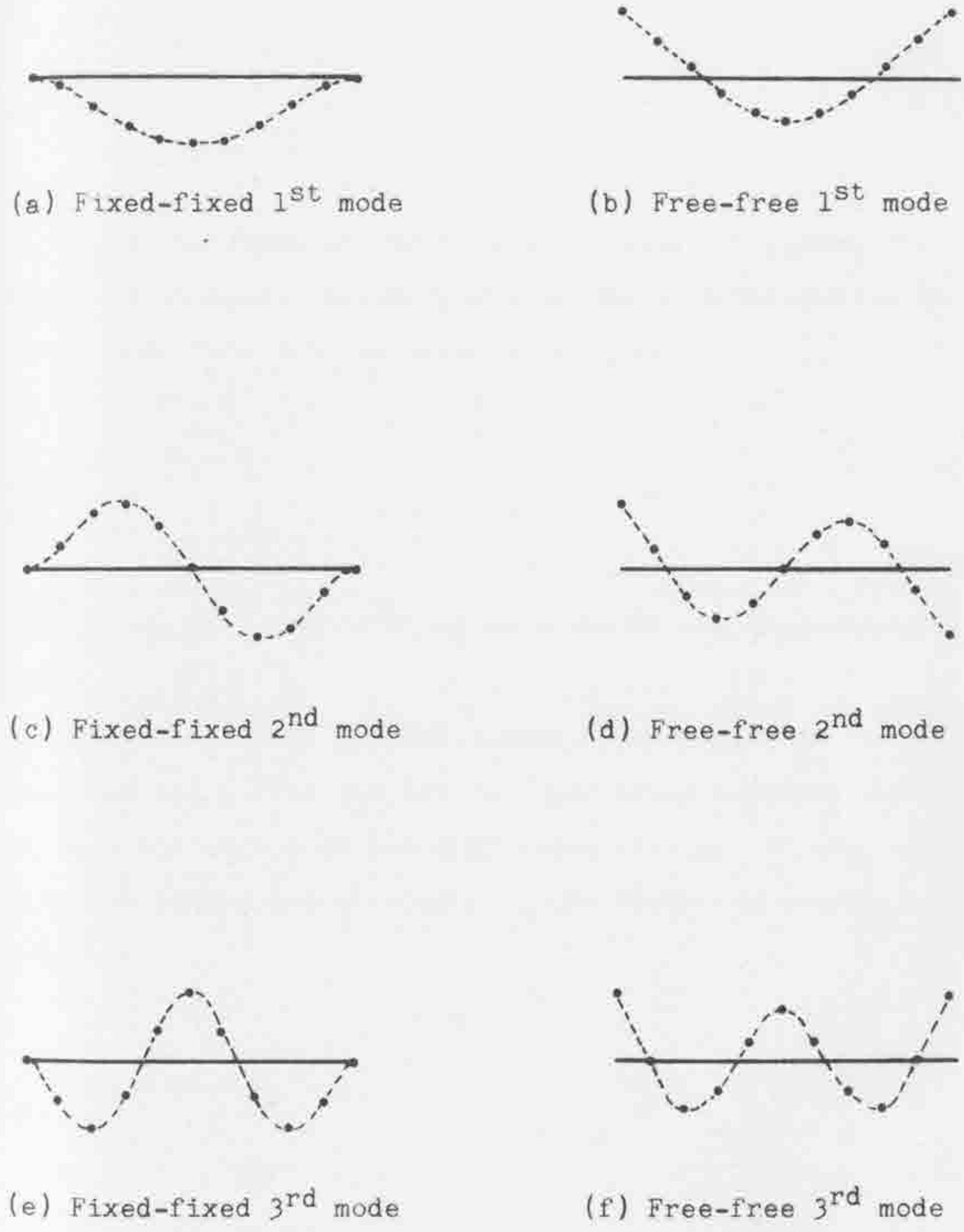


FIGURE 2.5 EIGENVALUE RATIO VERSUS EFFECTIVE CONNECTION FOR A FIXED-FIXED BEAM



(a) Fixed-fixed 1^{st} mode

(b) Free-free 1^{st} mode

(c) Fixed-fixed 2^{nd} mode

(d) Free-free 2^{nd} mode

(e) Fixed-fixed 3^{rd} mode

(f) Free-free 3^{rd} mode

FIGURE 2.6 TYPICAL MODE SHAPES

$$Y = 0 \ ,$$
$$Y'' = 0$$

and

$$F = 0 \ .$$

Now since the equation for F is applicable throughout the domain of interest including the boundary, substitution of these conditions into equation 2.58 yields

$$Y = 0 \ ,$$
$$Y'' = 0$$

and

$$Y^{IV} = 0 \tag{2.59}$$

for the equivalent conditions at a pinned end in terms of Y only.

In a completely similar manner, the conditions for a fixed end and a free end may be found using equation 2.58, and equation sets 2.26 and 2.27 respectively. If the indicated operations are carried out, the resulting conditions are

$$Y = 0 \ ,$$
$$Y' = 0$$

and

$$Y^{V} - 8KY''' = 0 \tag{2.60}$$

at a fixed end and

$$Y'' = 0 \ ,$$
$$Y^{IV} - N\omega^2 Y = 0$$

and

$$Y^{V} - 9KY''' - N\omega^2 Y' = 0 \qquad 2.61$$

at a free end.

The equivalence of these conditions is most easily seen by investigating a pinned-pinned beam. If the boundary conditions given by equation set 2.60 are used, the resulting equations are

$$[BC] \left\{a\right\} = \left\{0\right\}$$

where now

$$[BC] =$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -\gamma^2 & 0 & \alpha^2 & 0 & \beta^2 \\ 0 & \gamma^4 & 0 & \alpha^4 & 0 & \beta^4 \\ \sin(\gamma) & \cos(\gamma) & \sinh(\alpha) & \cosh(\alpha) & \sinh(\beta) & \cosh(\beta) \\ -\gamma^2\sin(\gamma) & -\gamma^2\cos(\gamma) & \alpha^2\sinh(\alpha) & \alpha^2\cosh(\alpha) & \beta^2\sinh(\beta) & \beta^2\cosh(\beta) \\ \gamma^4\sin(\gamma) & \gamma^4\cos(\gamma) & \alpha^4\sinh(\alpha) & \alpha^4\cosh(\alpha) & \beta^4\sinh(\beta) & \beta^4\cosh(\beta) \end{bmatrix}.$$

When the determinant of BC is now equated to zero the resulting equation is

$$[(\alpha^2 + \gamma^2)(\beta^4 - \gamma^4) - (\beta^2 + \gamma^2)(\alpha^4 - \gamma^4)]$$
$$\sin(\gamma)\sinh(\alpha)\sinh(\beta) = 0$$

which may be reduced to

$$(\beta^2 - \alpha^2)\ \sin(\gamma)\sinh(\beta)\sinh(\alpha) = 0. \qquad 2.62$$

Since α and β are distinct, equation 2.62 reduces to

$$\sin(\gamma)\sinh(\alpha)\sinh(\beta) = 0. \qquad 2.63$$

It is noted that equation 2.63 is the same as equation 2.48. Following the same procedure as in Section 2.6, the resulting eigenvalues and eigenvectors are

$$\gamma = n\pi$$

where

$$n = 1,2,3\ \ldots$$

and

$$Y = a_1 \sin(\gamma z)\ .$$

In a similar manner the other conditions were used in the computer programs and the results obtained will be the same as those of section 2.6.

CHAPTER 3

GENERALIZATION OF THE THEORY

3.1 Introduction

In this chapter the results of Chapter 2 are generalized to systems consisting of an arbitrary number of layers with a single plane of symmetry. The asymmetry can be considered to come from variation of the physical properties from layer to layer, or from variation of the connector modulus between joints. It can also arise naturally as with the asymmetry associated with T-beams. It is assumed that the plane of symmetry coincides with the plane of vibration. This assumption avoids the complications associated with a coupling of torsional effects with lateral effects.

Numerous examples are worked for the various types of systems covered. Additionally, it is shown that the equations developed have solutions which limit properly to expected results for a very high connector modulus.

Some discussion is devoted to the operator obtained for a general system because it differs substantially from what would be expected from a review of the work of Pleshkov (23) and Rassam (24).

A numerical procedure is presented to allow for the inclusion of the effects of variation along the length of the beam of both the layer and connector properties.

Finally, the inclusion of damping in the solution is considered and the limitations of the damped solution are discussed.

3.2 Two Layer System

The development for a two layer system is directly applicable to a T-beam. Additionally, it is applicable for the case where the two layers have different moduli of elasticity. This case is handled by using the transformed cross-section where a single modulus of elasticity is used with transformed widths.

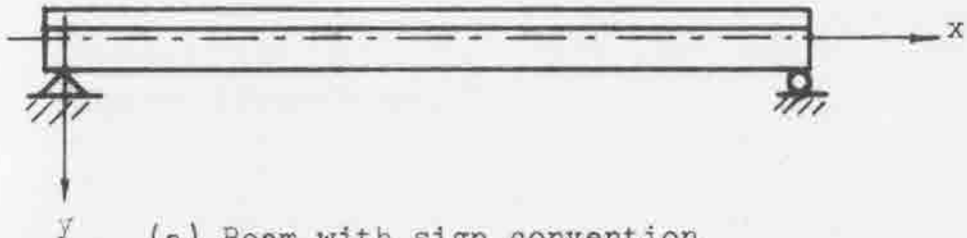
Figure 3.1 shows a T-beam with a transformed cross-section which is representative of a general two layer system. In this figure the following nomenclature is used:

$\overline{h}$ = the distance from the top of beam to the centroid of the transformed cross-section (in.).

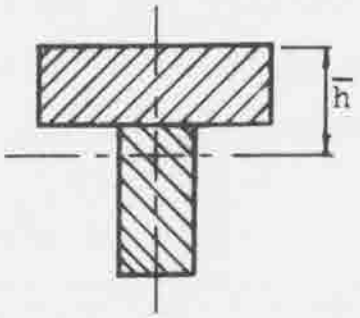
r_i = the distance from the system centroid to the centroid of the ith layer.

Now as with the three equal layer case, Newton's second law is applied to the beam element shown in Figure 3.1 (c). The resulting equations are

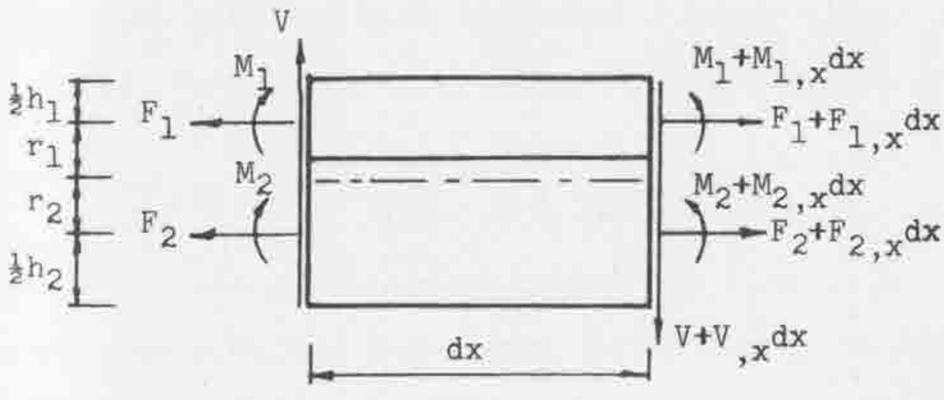
$$F_{1,x} + F_{2,x} = 0 \; , \qquad 3.1$$



(a) Beam with sign convention



(b) Cross-section



(c) Beam element

FIGURE 3.1 TWO LAYER SYSTEM

$$V_{,x} = \sum_{1}^{2} \rho_i A_i Y_{i,2t} \,, \tag{3.2}$$

where

A_i = untransformed area,

and

$$V = \sum_{1}^{2} (M_{i,x} + r_i F_{i,x}) \,, \tag{3.3}$$

where rotary inertia has been neglected. If it is now assumed that each layer deflects the same amount such that to first order each layer has the same curvature, then

$$M_i = -EI_i Y_{,2x}$$

and

$$Y_{i,2t} = Y_{,2t} \,.$$

With these substitutions equations 3.2 and 3.3 are combined to yield, for properties constant along the beam length,

$$\sum_{1}^{2} EI_i Y_{,4x} - \sum_{1}^{2} r_i F_{i,2x} + \sum_{1}^{2} \rho_i A_i Y_{,2t} = 0 \,. \tag{3.4}$$

It is now necessary to relate the F_i's to Y. Now proceeding as with the three equal layer case, the displacement difference between the first and second layer is

$$\Delta S_{21} = \frac{S}{kn} F_{2,x} \tag{3.5}$$

and

$$\Delta S_{21} = \int_{o}^{x} \epsilon_2^{u} dx - \int_{o}^{x} \epsilon_1^{L} dx \tag{3.6}$$

where for tension taken as positive,

ϵ_2^u = the strain at the upper surface of the second layer,

$$= \frac{F}{EA_2^*} + \frac{h_2}{2} Y_{,2x}$$

and

ϵ_1^L = the strain at the lower surface of the first layer,

$$= \frac{F_1}{EA_1^*} - \frac{h_1}{2} Y_{,2x} ,$$

with

A_1^* = the transformed area of the ith layer.

Equations 3.5 and 3.6 are now combined and differentiated once to give

$$\frac{S}{kn} F_{2,2x} = \frac{F_2}{EA_2^*} - \frac{F_1}{EA_1^*} + \left(\frac{h_1 + h_2}{2}\right) Y_{,2x} \; . \qquad 3.7$$

Equations 3.1, 3.4, and 3.7 represent a system of three equations in three unknowns and are the governing equations for the general two layer system. The set of equations is reduced to two by using horizontal equilibrium (equation 3.1). By first noting from the definition of the centroid

$$r_2 - r_1 = \frac{h_1 + h_2}{2}$$

and eliminating F_2, then

$$\sum_1^2 EI_i Y_{,4x} + (r_2 - r_1)F_{1,2x} + \sum_1^2 \rho_i A_i Y_{,2t} = 0 \qquad 3.8$$

and

$$\frac{S}{kn}F_{1,2x} - \frac{1}{E}\left(\frac{1}{A_1^*} + \frac{1}{A_2^*}\right)F_1 + (r_2 - r_1)Y_{,2x} = 0. \qquad 3.9$$

As with the three equal layer case, there are six boundary conditions necessary for the complete solution to equations 3.8 and 3.9. From an inspection of Figure 3.1 (c) it is seen that the end moment may be written as

$$M_T = M_1 + M_2 - (h_2 + \frac{h_1}{2})F_1 - \frac{h_2}{2} F_2$$

which may be simplified to

$$M_T = -\sum_1^2 EIY_{,2x} - (r_2 - r_1)F_1 \ . \qquad 3.10$$

similarly

$$V_T = -\Sigma EIY_{,3x} - (r_2 - r_1)F_{1,x} \ . \qquad 3.11$$

Now following the reasoning used for the case of three equal layers, the boundary conditions may be written as

$$\begin{aligned} Y &= 0 \ , \\ Y_{,2x} &= 0 \ , \\ F_1 &= 0 \ , \end{aligned} \qquad 3.12$$

at a simply supported end,

$$Y = 0\ ,$$
$$Y_{,x} = 0\ ,$$
$$F_{1,x} = 0\ , \tag{3.13}$$

at a fixed end and

$$Y_{,2x} = 0\ ,$$
$$F_1 = 0\ ,$$
$$V_T = 0\ , \tag{3.14}$$

at a free end.

Equations 3.8 and 3.9 are linear partial differential equations with constant coefficients and the time dependency may be separated out by assuming

$$Y(x,t) = Y(x)\cos(\omega t + \varphi)$$

$$F_1(x,t) = F_1(x)\cos(\omega t + \varphi)\ .$$

If the above substitutions are made and the necessary manipulations are performed such that all beams are of length unity, the governing equations given by 3.8 and 3.9 may be placed in matrix operator form as

$$\begin{bmatrix} \frac{\Sigma EI}{L^4} D^4 - \Sigma \rho A \omega^2 & \left(\frac{r_2 - r_1}{L^2}\right) D^2 \\ \frac{(r_2 - r_1)}{L^2} D^2 & \frac{S}{knL^2} D^2 - \frac{1}{E}\left(\frac{1}{A_1^*} + \frac{1}{A_2^*}\right) \end{bmatrix} \begin{Bmatrix} Y \\ F_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3.15}$$

where

$$D^n = \frac{d^n}{dz^n} .$$

The characteristic equation for Y or F_i may now be found by setting the determinant of the operator matrix equal to zero. This yields for Y after some algebraic manipulations

$$Y^{VI} - K\left(1 + \frac{(r_2 - r_1)^2}{\Sigma I\left(\frac{1}{A_1^*} + \frac{1}{A_2^*}\right)}\right) Y^{IV} - N\omega^2 Y'' + KN\omega^2 Y = 0 \qquad 3.16$$

where

$$K = \frac{knL^2}{SE}\left(\frac{1}{A_1^*} + \frac{1}{A_2^*}\right)$$

and

$$N = \sum_1^2 \left(\frac{\rho_i A_i L^4}{EI_i}\right) .$$

Consider now a solid beam like that shown in Figure 3.1 (b). The moment of inertia about the centroidal axis may be written as

$$I_s = \sum_1^2 I_i + A_1^* r_1^2 + A_2^* r_2^2$$

$$= \Sigma I_i (T.C.)$$

where the transfer constant T.C. is defined as

$$T.C. = 1 + \frac{A_1^* r_1^2 + A_2^* r_2^2}{\Sigma I_i} . \qquad 3.17$$

The bracketed coefficient of Y^{IV} in equation 3.16 may be expanded as

$$1 + \frac{1}{\Sigma I_i} (r_1^2 - 2r_1 r_2 + r_2^2) \left(\frac{A_1^* A_2^*}{A_1^* + A_2^*}\right) .$$

It is noted from the definition of the centroid, with the sign convention chosen, that

$$r_2 A_2^* = - r_1 A_1^* .$$

The coefficient may then be written as

$$1 + \frac{1}{\Sigma I_i} \left(\frac{r_1^2 A_1^*(A_1^* + A_2^*) + r_2^2 A_2(A_1^* + A_2^*)}{A_1^* + A_2^*}\right)$$

or

$$T.C. = 1 + \frac{1}{\Sigma I_i} (r_1^2 A_1^* + r_2^2 A_2^*) .$$

Equation 3.16 then becomes

$$Y^{VI} - (T.C.)KY^{IV} - N\omega^2 Y'' + KN\omega^2 = 0 . \qquad 3.18$$

The similarity between equation 3.18 and equation 2.33 for the three equal layer case is evident. The solution to equation 3.18 follows the solution method shown in Chapter 2.

For convenience it is desired to write the solution to equation 3.18 in the same functional form as that of the three equal layer system. To do this, it is necessary to show that the discriminant of the third degree polynomial which results from the assumed solution is less than zero.

Following the procedure shown in Section 2.5, the discriminant for the two layer case is found to be

$$\text{Dis} = -\frac{N\omega^2}{27}\left[N^2\omega^4 + N\omega^2K^2(T.C.)^2\left(\frac{1}{4} + \frac{9}{2(T.C.)} - \frac{27}{4(T.C.)^2}\right)\right.$$

$$\left. + K^4(T.C.)^4\left(\frac{1}{6} + \frac{1}{2(T.C.)}\right)\right] . \qquad 3.19$$

From an inspection of equation 3.17 it is seen that T.C. is always greater than one. Suppose T.C. is chosen as 1.05. Then equation 3.19 can then be written as

$$\text{Dis} = -\frac{N\omega^2}{27}\left[(N\omega^2 - .97K^2)^2 + .3N\omega^2K^2\right]$$

which will always be less than zero for ω real and K and $N > 0$. It is clear that for all T.C. greater than 1.05 the discriminant as indicated by equation 3.19 is less than zero.

The choice of T.C. equal to 1.05 represents no limitation on the problem because it is indicative of a beam with one area at least 60 times the other. This type of beam is of no practical significance. Thus the solution to equation 3.18 can be written as

$$Y = a_1\sin(\gamma z) + a_2\cos(\gamma z) + a_3\sinh(\alpha z) + a_4\cosh(\alpha z)$$

$$+ a_5\sinh(\beta z) + a_6\cosh(\beta z) \qquad 3.20$$

where γ^2, α^2, and β^2 are the roots to the equation

$$p^3 - (T.C.)Kp^2 - N\omega^2p + KN\omega^2 = 0 . \qquad 3.21$$

The eigenvalues and eigenvectors can now be found for the set of boundary conditions of interest exactly as shown in Chapter 2. If, for example, the simply supported boundary conditions are chosen and the procedure is followed, the values of ω^2 are given by

$$\omega^2 = \frac{(n\pi)^4}{N} \left(\frac{(n\pi)^2 + (T.C.)K}{(n\pi)^2 + K}\right) . \qquad 3.22$$

If K is very large as when the layers are rigidly connected, equation 3.22 then reduces to

$$\omega^2 = \frac{(T.C.)(n\pi)^4}{N} .$$

Now solving for ω by noting that

$$\Sigma\rho_i A_i^* = (\rho A)_s$$

and

$$(\Sigma I_i)(T.C.) = I_s$$

gives

$$\omega = (n\pi)^2 \sqrt{\frac{EI_s}{(\rho A)_s L^4}} . \qquad 3.23$$

Equation 3.23 represents the solution for the natural frequency of a solid beam with simply supported boundary conditions and shows that the solution limits properly for rigid connection.

For negligible K the limiting solution is

$$\omega = (n\pi)^2 \sqrt{\frac{\Sigma EI_i}{\Sigma\rho_i A_i^* L^4}} . \qquad 3.24$$

The physical achievement of this lower limit is open to question. This is because for no interlayer connection, each layer will tend to vibrate at its own natural frequency. Since these frequencies are different, the assumption that the layers always remain in contact will be violated.

Equations 3.22 and 3.23 may be used to obtain a useful set of curves. First the following definition is made

$$\xi = \frac{\omega^2}{\omega_s^2} \qquad 3.25$$

where

ω_s = the natural frequency of a solid beam of the same cross-section as the two layer system.

Substitution now yields

$$\xi = \frac{\left(K + \frac{(n\pi)^2}{T.C.} \right)}{K + (n\pi)^2} \qquad 3.26$$

Figure 3.2 shows a plot of ξ versus K for various values of T.C. for the first mode. As an example of the use of these curves, consider a T-beam made of two wood 2 x 4 sections. The following properties are assumed:

$E_1 = E_2 = 10^6$ lb/in.^{2}	$k = 3\times10^4$ lb/in/connector.
$b_1 = h_2 = 3.5$ in.	$L = 180$ in.
$b_2 = h_1 = 1.5$ in.	$n = 1.0$
$\rho = 4.5\times10^{-4}$ slugs/in.^{3}	$s = 3$ in.

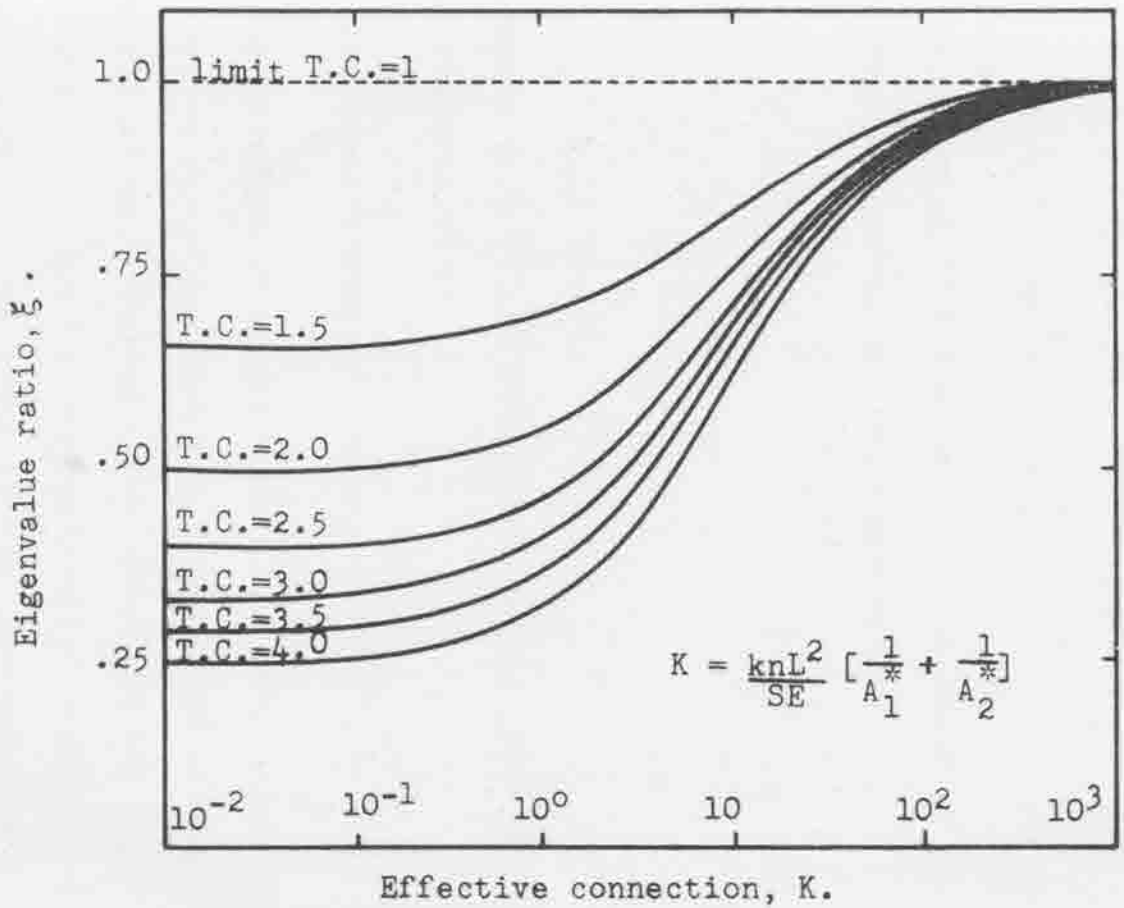


FIGURE 3.2 EIGENVALUE RATIO VERSUS EFFECTIVE CONNECTION FOR A SIMPLY SUPPORTED BEAM WITH VARIOUS TRANSFER CONSTANTS

From these data the constants of the problem are calculated as

$$T.C. = 3.68,$$
$$K = 4.93$$

and

$$N = 2.51 \times 10^{-2}.$$

Now interpolating from Figure 3.2, ξ is found to be .525. The natural frequency is then computed as

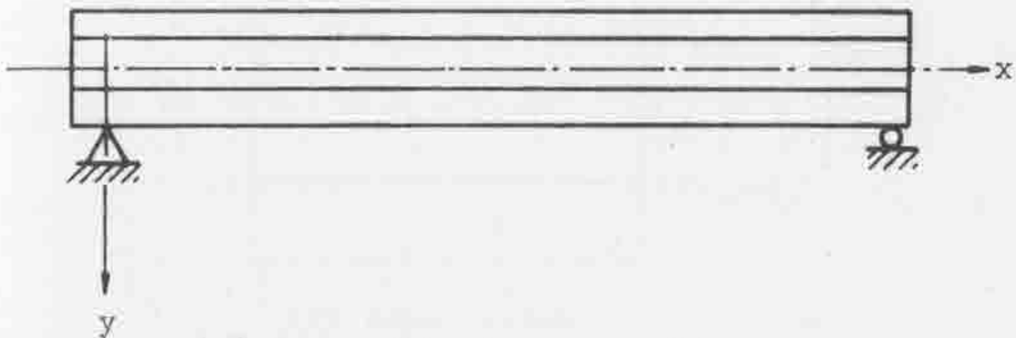
$$\omega = \sqrt{\xi}\,\omega_s ,$$
$$= \pi^2 \sqrt{\frac{(\xi)(T.C.)}{N}} ,$$
$$= 13.9 \text{ hertz},$$

as compared with 10 and 19.2 hertz at the lower and upper limits respectively.

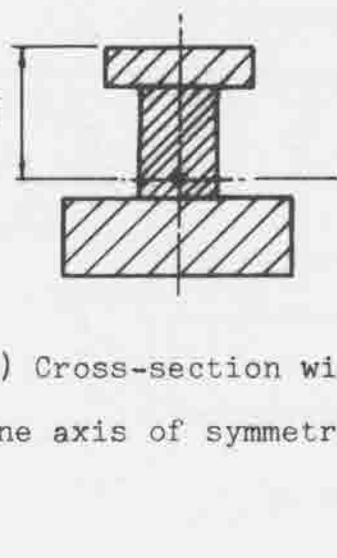
3.3 Three Layer System

The development of the equations for a general three layer system directly parallels that of the two layer system. Figure 3.3 shows a typical three layer system. Again, transformed widths are used to compensate for modulous of elasticity differences. Referring to the beam element, Newton's second law is applied such that the following three equations are obtained:

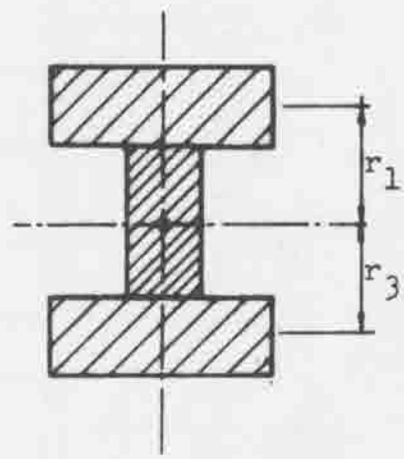
$$\sum_{1}^{3} F_{i,x} = 0 , \qquad 3.27$$



(a) Beam with sign convention

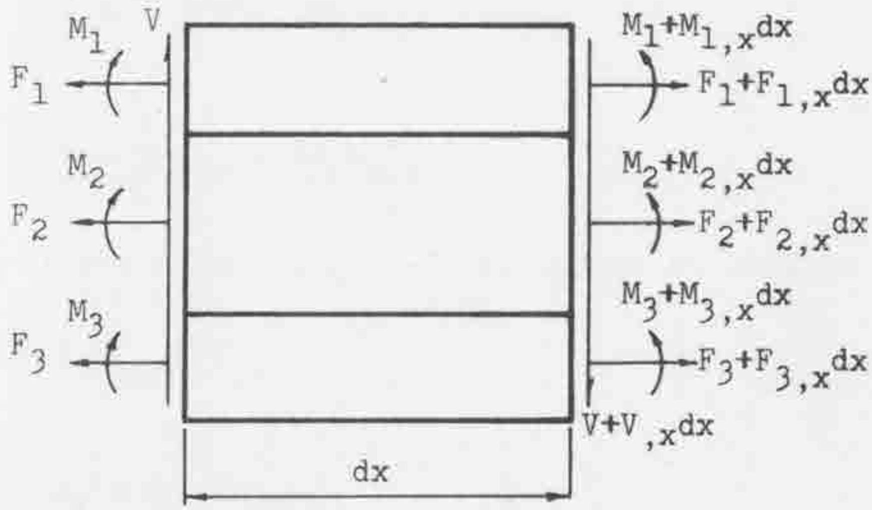


(b) Cross-section with one axis of symmetry

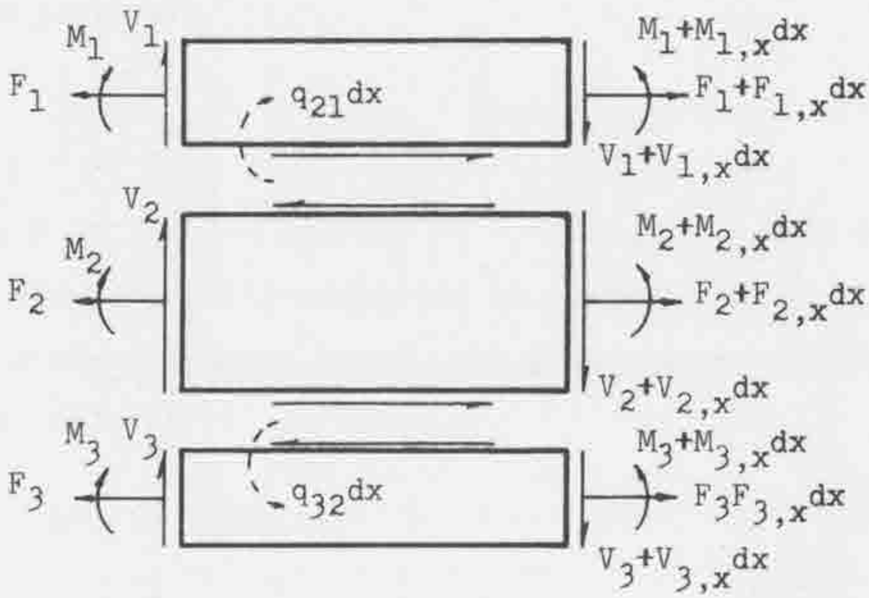


(c) Dual symmetric cross-section

FIGURE 3.3 THREE LAYER SYSTEM



(d) Beam element



(e) Layer elements

FIGURE 3.3--Continued

$$V_{,x} = \sum_1^3 \rho_i A_i Y_{,2t} \qquad 3.28$$

and

$$V = \sum_1^3 M_{i,x} + \sum_1^3 r_i F_{i,x} \; . \qquad 3.29$$

For the layers deflecting the same amount it is seen that

$$Y_{i,2t} = Y_{,2t}$$

and

$$M_i = -EI_i Y_{,2x} \; .$$

With these substitutions, equations 3.28 and 3.29 may now be combined to give

$$\sum_1^3 EI_i Y_{,4x} - \sum_1^3 r^i F_{i,2x} + \sum_1^3 \rho_i A_i Y_{,2t} = 0 \; . \qquad 3.30$$

The F_i's are now related to Y through the slip equations. Since the slip permitted by a connector is proportional to the connector force, the displacement differences may be written as

$$\Delta S_{21} = \left(\frac{S}{kn}\right)_{21} q_{21}$$

and

$$\Delta S_{32} = \left(\frac{S}{kn}\right)_{32} q_{32}$$

These relationships may be rewritten using horizontal equilibrium of the layer elements as shown in Figure 3.3 (e). This gives

$$\Delta S_{21} = -\left(\frac{S}{kn}\right)_{21} F_{1,x}$$

and

$$\Delta S_{32} = (\frac{S}{kn})_{32} F_{3,x} \ . \qquad 3.31$$

The displacement differences may also be written in integral form as

$$\Delta S_{21} = \int_{o}^{x} \epsilon_2^u dx - \int_{o}^{x} \epsilon_1^L dx$$

$$\Delta S_{32} = \int_{o}^{x} \epsilon_3^u dx - \int_{o}^{x} \epsilon_2^L dx \qquad 3.32$$

Proceeding as with the two layer case, with tension again taken as positive, equation sets 3.31 and 3.32 are combined to yield

$$-(\frac{S}{kn})_{12} F_{1,2x} = \frac{F_2}{EA_2^*} - \frac{F_1}{EA_1^*} + (\frac{h_1 + h_2}{2}) Y_{,2x} \qquad 3.33$$

and

$$(\frac{S}{kn}) F_{3,2x} = \frac{F_3}{EA_3^*} - \frac{F_2}{EA_1^*} + (\frac{h_2 + h_3}{2}) \ Y_{,2x} \qquad 3.34$$

Equations 3.27, 3.30, 3.33, and 3.34 form a set of four equations in four unknowns and are the governing equations of the general three layer system. The set may be reduced to three equations in three unknowns by using equation 3.27 to eliminate one of the F_i's. If F_3 is eliminated the governing equations may be written as

$$\sum_1^3 EI_i Y_{,4x} + \sum_1^2 (r_3 - r_i) F_{i,2x} + \sum_1^3 \rho_i A_i Y_{,2t} = 0 \ ,$$

$$\left(\frac{S}{kn}\right)_{12} F_{1,2x} - \frac{F_1}{EA_1^*} + \frac{F_2}{EA_2^*} + \left(\frac{h_1 + h_2}{2}\right) Y_{,2x} = 0$$

and

$$\left(\frac{S}{kn}\right)_{32} F_{1,2x} - \frac{F_1}{EA_3} + \left(\frac{S}{kn}\right)_{32} F_{2,2x} - \left(\frac{1}{EA_2^*} + \frac{1}{EA_3^*}\right) F_2$$

$$+ \left(\frac{h_3 + h_2}{2}\right) Y_{,2x} = 0 \ . \qquad 3.35$$

The boundary conditions for the three layer problem are now found in the same manner as with the two layer problem. A total of eight boundary conditions are now necessary. From Figure 3.3 (d), the end moment is written as

$$M_T = \sum_1^3 M_i - (h_3 + h_2 + \frac{h_1}{2}) F_1 - (h_3 + \frac{h_2}{2}) F_2 - \frac{h_3}{2} F_3 \ .$$

Which may be rewritten as

$$M_T = -\sum_1^3 EI_i Y_{,2x} - (\frac{h_1}{2} + h_2 + \frac{h_3}{2}) F_1 - (\frac{h_2}{2} + \frac{h_3}{2}) F_2 \ .$$

Similarly

$$V_T = -\Sigma EI_i Y_{,3x} - (\frac{h_1}{2} + h_2 + \frac{h_3}{2}) F_{1,x} - (\frac{h_2}{2} + \frac{h_3}{2}) F_{2,x} \ .$$

Now following the previous reasoning, the boundary conditions may be written as

$$Y = 0 ,$$
$$Y_{,2x} = 0 ,$$
$$F_1 = 0 ,$$
$$F_2 = 0 , \qquad 3.36$$

at a pinned end,

$$Y = 0 ,$$
$$Y_{,x} = 0 ,$$
$$F_{1,x} = 0 ,$$
$$F_{2,x} = 0 , \qquad 3.37$$

at a fixed end and

$$Y_{,2x} = 0 ,$$
$$F_1 = 0 ,$$
$$F_2 = 0 ,$$
$$V_T = 0 , \qquad 3.38$$

at a free end.

The solution technique for the three layer system is essentially the same as that for the systems encountered previously. The main difference is only one of added algebraic detail. Once again simple harmonic motion may be assumed. The governing equations can be written in linear operator form. From the definition of the centroid, it is noted that

$$r_3 - r_1 = \frac{h_1 + h_2}{2} + \frac{h_2 + h_3}{2}$$

and

$$r_3 - r_2 = \frac{h_3 + h_2}{2} \quad .$$

Equation set 3.35 can now be written for beams of arbitrary length as

$$\left[\, L \,\right] \begin{Bmatrix} Y \\ F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

where

$$[L] =$$

$$\begin{bmatrix} \frac{\Sigma EI_i D^4}{L^4} - \Sigma\rho A\omega^2 & (C_{12} + C_{23})D^2 & C_{23}D^2 \\ C_{12}D^2 & G_{21}D^2 - T_1 & T_2 \\ C_{23}D^2 & G_{32}D^2 - T_3 & G_{32}D^2 - T_2 - T_3 \end{bmatrix}$$

with

$$C_{ij} = \frac{h_i + h_j}{2L^2} \quad ,$$

$$G_{ij} = \left(\frac{S}{knL^2}\right)_{ij}$$

and

$$T_i = \frac{1}{EA_i^*}$$

The linear operator which provides the characteristic equation for Y or F is now found by taking

$$\left| \; L \; \right| = 0.$$

This operation yields after some algebraic manipulations

$$\left(\frac{\Sigma EI}{L^4} G_{21}G_{32}\right)D^8 - \left\{\left(G_{32}T_1 \left[1 + \frac{C_{12}^2 L^4}{T_1 \Sigma EI}\right] + G_{21}T_3 \left[1 + \frac{C_{32}^2 L^4}{T_3 \Sigma EI}\right]\right.\right.$$

$$\left.\left. + (G_{21} + G_{32})T_2\right) \frac{\Sigma EI}{L^4} \right\} D^6 + \left\{ \left(\frac{\Sigma EI}{L^4}\right)(T_1T_2 + T_2T_3 + T_3T_1)\right.$$

$$\left. + C_{12}^2T_3 + (C_{12} + C_{23})^2T_2 + C_{23}^2T_1 - (\Sigma\rho A\omega^2)(G_{21}G_{32}) \right\} D^4$$

$$+ \left\{ (\Sigma\rho A\omega^2)(G_{32}T_1 + (G_{21} + G_{32})T_2 + G_{12}T_3) \right\} D^2 - \left\{(\Sigma\rho A\omega^2)\right.$$

$$\left.(T_1T_2 + T_2T_3 + T_3T_1)\right\} , \qquad 3.39$$

or

$$D^8 - \left\{ \frac{T_1}{G_{21}} \left[1 + \frac{C_{12}^2 L^4}{T_1 \Sigma EI}\right] + \frac{T_3}{G_{32}} \left[1 + \frac{C_{23}^2 L^4}{T_3 \Sigma EI}\right] + \frac{(G_{21}+G_{32})}{G_{21}G_{32}} T_2 \right\} D^6$$

$$+ \left\{ \frac{T_1T_2 + T_2T_3 + T_3T_1}{G_{21}G_{32}} + \frac{C_{12}^2T_3 + (C_{12} + C_{23})^2T_2 + C_{23}^2T_1}{G_{21}G_{32}\frac{\Sigma EI}{L^4}}\right.$$

$$\left. - \frac{\Sigma\rho A\omega^2 L^4}{\Sigma EI} \right\} D^4 + \left\{ \left[\frac{T_1}{G_{21}} + \frac{(G_{21} + G_{32})}{G_{21}G_{32}} T_2 + \frac{T_3}{G_{32}}\right]\left[\frac{\Sigma\rho A\omega^2 L^4}{\Sigma EI}\right] \right\} D^2$$

$$- \left\{ \left[\frac{T_1T_2 + T_2T_3 + T_3T_1}{G_{21}G_{32}}\right]\left[\frac{\Sigma\rho A\omega^2 L^4}{\Sigma EI}\right] \right\} . \qquad 3.40$$

The linear operator given by 3.40 will provide an eighth order equation in Y or F where as before only a sixth order equation was provided. The higher order is due to the fact that there are now two F's involved in the equations where only one was involved in the previous cases. The order of the operator should reduce to six for the special cases of three equal layers, a dual symmetric system, and a system where one interlayer connection gets very large.

The first case to be considered will be that of a system of three equal layers. For this case

$$G_{ij} = G \ ,$$
$$T_i = T \ ,$$
$$C_{ij} = \frac{h}{L^2} \ ,$$
$$I_i = I \ ,$$
$$A_i = A \ ,$$

and

$$A_i^* = A \ .$$

The operator given by 3.40 then reduces to

$$D^8 - \frac{T}{G}\left\{2[1 + \frac{h^2}{3EIT}] + 2\right\}D^6 + \left\{3(\tfrac{T}{G})^2[1 + \frac{2h^2}{3EIT}] - \frac{\rho A\omega^2 L^4}{EI}\right\}D^4$$
$$+ \left\{\frac{4T}{G}\,\frac{\rho A\omega^2 L^4}{EI}\right\}D^2 - \left\{3(\tfrac{T}{G})^2\,\frac{\rho A\omega^2 L^4}{EI}\right\}. \qquad 3.41$$

Noting that

$$\frac{h^2}{3EIT} = 4$$

and letting

$$\frac{T}{G} = \frac{knL^2}{SEA} = K$$

and

$$N = \frac{\rho AL^4}{EI}$$

gives

$$D^8 - 12KD^6 + (27K^2 - N\omega^2)D^4 + 4KN\omega^2D^2 - 3K^2N\omega^2 \ .$$

The above operator may be factored into the form

$$(D^2 - 3K)(D^6 - 9KD^4 - N\omega^2D^2 + KN\omega^2) \ . \qquad 3.42$$

The first part of 3.42 will provide solutions independent of ω, of the form

$$Y = a \sinh(\sqrt{3K}\ z) + b \cosh(\sqrt{3K}\ z) \ .$$

Solutions of this form can satisfy the various boundary conditions only for

$$a = b = 0 \ .$$

Then for a nontrivial solution, 3.42 reduces to

$$D^6 - 9KD^4 - N\omega^2D^2 + KN\omega^2 \ . \qquad 3.43$$

The operator given by 3.43 when applied to Y gives

$$Y^{VI} - 9KY^{IV} - N\omega^2Y'' + KN\omega^2Y = 0 \ . \qquad 3.44$$

Equation 3.44 is equivalent to equation 2.33 developed previously. This indicates that the general three layer operator limits properly for the special case of three equal layers.

Now consider a dual symmetric system such as an I-beam. This system should also have a sixth order equation since a symmetry argument can be used to show that F_2 is zero. For the dual symmetric system

$$G_{ij} = G \ ,$$

$$A_1 = A_3 = A \ ,$$

$$T_1 = T_3 = T$$

and

$$C_{12}^2 = C_{23}^2 = \frac{r^2}{L^2} \ .$$

Making the above substitutions into 3.40 and factoring yields

$$[D^2 - \frac{(2T_2 + T)}{G}][D^6 - \frac{T}{G}(1 + \frac{2r^2}{T\Sigma EI})D^4 - \frac{\Sigma\rho AL^4\omega^2}{\Sigma EI} D^2 + \frac{T}{G}\frac{\Sigma\rho AL^4\omega^2}{\Sigma EI}] \ . \qquad 3.45$$

The first term of 3.45 will provide solutions of the form

$$Y = a \sinh(\sqrt{\frac{2T_2+T}{G}}\, z) + b \cosh(\sqrt{\frac{2T_2+T}{G}}\, z) \ ,$$

which again will satisfy the boundary conditions only for

$$a = b = 0 \ .$$

Thus for non-trivial solutions the linear operator is reduced to sixth order. The reduced operator, operating on Y, yields the following equation

$$Y^{VI} - (\frac{T}{G})(1 + \frac{2r^2}{T\Sigma EI})Y^{IV} - \frac{\Sigma\rho AL^4}{\Sigma EI}\omega^2 Y''$$

$$+ (\frac{T}{G})(\frac{\Sigma\rho AL^4}{\Sigma EI}\omega^2)Y = 0 \; . \qquad 3.46$$

The moment of inertia of a solid beam with a dual symmetric cross-section as shown in Figure 3.3 (c) may be written as

$$I_s = \Sigma I + A_1 r_1^2 + A_3 r_3^2$$

or since

$$r_1^2 = r_3^2 = r^2$$

and

$$A_1 = A_3 = A$$

then

$$I_s = \Sigma I(T.C.) = \Sigma I(1 + \frac{2Ar^2}{\Sigma I})$$

where T.C. is a transfer constant.

If it is noted that

$$1 + \frac{2r^2}{T\Sigma EI} = 1 + \frac{2Ar^2}{\Sigma I}$$

then equation 3.46 may be written as

$$Y^{VI} - (T.C.)KY^{IV} - N\omega^2 Y'' + KN\omega^2 Y = 0 \qquad 3.47$$

where

$$K = \frac{knL^2}{SEA}$$

and

$$N = \frac{\Sigma\rho AL^4}{\Sigma EI}$$

The solution to equation 3.47 will follow exactly the procedure outlined in Section 3.2. Following the reasoning of that section, it is immediately evident that the solution to 3.47 will reduce that of a solid Euler beam as K gets very large.

The case of letting one interlayer connection approach infinity will now be considered. This is equivalent to a two layer system and the characteristic equation should be sixth order. Suppose that k_{12} approaches infinity. Then G_{21} approaches zero and the operator given by 3.39 reduces to

$$- \left\{ (G_{32}T_1[1 + \frac{C_{12}^2 L^4}{T_1 \Sigma EI}] + G_{32}T_2) \frac{\Sigma EI}{L_4} \right\} D^6 + \left\{ (\frac{\Sigma EI}{L_4})(T_1T_2 + T_2T_3 + T_3T_1) + C_{12}^2 T_3 + (C_{12} + C_{23})^2 \, T_2 + C_{23}^2 T_1 \right\} D^4 + \left\{ (\Sigma\rho A\omega^2)(G_{32}T_1 + G_{32}T_2) \right\} D^2 - \left\{ (\Sigma\rho A\omega^2)(T_1T_2 + T_2T_3 + T_3T_1) \right\} . \qquad 3.48$$

The operator given by 3.48 is indeed sixth order, but is algebraically complicated. To avoid too much algebraic detail, consider the case when the layers started as equal layers. Then 3.48 may be written as

$$- \frac{3EIGT}{L^4} [2 + \frac{h^2}{3EIT}]D^6 + (\frac{9EIT^2}{L^4} + 6h^2T)D^4$$

$$+ 6GT\rho A\omega^2 D^2 - 9T^2\rho A\omega^2 \ . \qquad 3.49$$

If the operator is now applied to Y and the definitions of T and G are used and noting that

$$\frac{h^2}{3EIT} = 4 \ ,$$

then the resulting equation is

$$Y^{VI} - \frac{9knL^2}{2SEA} Y^{IV} - \frac{\rho AL^4}{3EI}\omega^2 Y'' + \frac{knL^2}{2SEA}\frac{\rho AL^4}{EI}\omega^2 Y = 0 \ . \qquad 3.50$$

Now consider a two layer system with equal bases but with

$$A_1 = 2A_2 = 2A \ .$$

For a two layer system with these areas, the problem constants may be calculated. These were given in Section 3.2 as

$$K = \frac{knL^2}{SE}\left(\frac{1}{A} + \frac{1}{2A}\right) = \frac{3knL^2}{2SEA} \ ,$$

$$N = \frac{3\rho AL^4}{9EI} = \frac{\rho AL^4}{3EI}$$

and

$$\text{T.C.} = 1 + \frac{(2A)(\frac{h}{2})^2 + A(h)^2}{9\,\frac{Ah^2}{12}} = 3 \ .$$

Now substitution into the general governing equation for a two layer system yields

$$Y^{VI} - \frac{9knL^2}{2SEA} Y^{IV} - \frac{\rho AL^4}{3EI} \omega^2 Y'' + \frac{knL^2}{2SEA} \frac{\rho AL^4}{EI} \omega^2 Y = 0 \ . \qquad 3.51$$

The equivalence of equations 3.50 and 3.51 shows that the three layer system properly reduces to a two layer system as one connector modulus gets very large.

The solution to the general three layer problem parallels the previous solution procedures except for detail. If the operator given by 3.40 is applied to Y, the resulting equation is

$$Y^{VIII} - \text{con1}Y^{VI} + (\text{con2} - N\omega^2)Y^{IV} + (\text{con3})(N\omega^2)Y''$$

$$- (\text{con4})(N\omega^2)Y = 0 \qquad 3.52$$

where

$$\text{con1} = \frac{T_1}{G_{21}} [1 + \frac{C_{12}^2 L^4}{T_1 \Sigma EI}] + \frac{T_3}{G_{32}} [1 + \frac{C_{23}^2 L^4}{T_3 \Sigma EI}] + T_2 \frac{(G_{21} + G_{32})}{G_{21}G_{32}},$$

$$\text{con2} = \frac{T_1T_2 + T_2T_3 + T_3T_1}{G_{21}G_{32}} + (\frac{L^4}{G_{21}G_{32}\Sigma EI})$$

$$(C_{12}^2 T_3 + (C_{12} + C_{23})^2 T_2 + C_{23}^2 T_1) \ ,$$

$$\mathrm{con3} = \frac{T_1}{G_{21}} + \frac{(G_{21} + G_{32})}{G_{21}G_{32}} T_2 + \frac{T_3}{G_{32}} ,$$

$$\mathrm{con4} = \frac{T_1T_2 + T_2T_3 + T_3T_1}{G_{21}G_{32}}$$

and

$$N = \frac{\Sigma \rho A L^4}{\Sigma EI}$$

Now proceeding as before, the solution is assumed in the form

$$Y = \sum_{1}^{8} \bar{a}_i e^{d_i z} . \qquad 3.53$$

This solution when applied to equation 3.52 yields an eighth order polynomial equation in d which is written as

$$d^8 - (\mathrm{con1})d^6 + (\mathrm{con2} - N\omega^2)d^4 + (\mathrm{con3})(N\omega^2)d^2 - (\mathrm{con4})(N\omega^2) = 0 . \qquad 3.54$$

It is noted that only 2n powers of d exist in equation 3.54. Thus, the polynomial may be reduced by making the substitution

$$p = d^2 .$$

The resulting polynomial is

$$p^4 - (\mathrm{con1})p^3 + (\mathrm{con2} - N\omega^2)p^2 + (\mathrm{con3})(N\omega^2)p - (\mathrm{con4})(N\omega^2) = 0 . \qquad 3.55$$

The solution for a particular set of boundary conditions may now be found by using the methods shown previously. Using the solution form of equation 3.53, a boundary condition matrix is formed. A value of ω is then assumed and the roots are computed using equation 3.55. The determinant of the boundary condition matrix is then computed. The zero's of a plot of the value of the determinant versus ω provide the eigenvalues of the problem.

The easiest eigenvalues to compute are those for a simply supported beam. From an inspection of equation 3.55, it is seen that regardless of whether con2 is greater than $n\omega^2$ or not, there are three sign changes in the polynomial equation. This indicates that three of the roots to the equation have positive real parts. Since all the constants are real, the roots must be real or occur in conjugate pairs. Since only one root has a negative real part it must then be real. Denoting this root as γ^2, it is easily shown that the solution for a simply supported beam is

$$Y = A\sin(\gamma z) \qquad 3.56$$

where

$$\gamma^2 = -(n\pi)^2$$

with

$$n = 1,2,3 \;....$$

With this information and equation 3.55, the values of ω^2 which are of interest may be computed. Solving for ω^2 yields

$$\omega^2 = \frac{(n\pi)^4}{N} \left(\frac{(n\pi)^4 + \text{con1}(n\pi)^2 + \text{con2}}{(n\pi)^4 + \text{con3}(n\pi)^2 + \text{con4}}\right) . \qquad 3.57$$

As a check on the validity of the above solution, consider the case when both connector moduli get very large. For this case, equation 3.57 reduces to

$$\omega^2 = \frac{(n\pi)^4}{N} \frac{\text{con2}}{\text{con4}} \qquad 3.58$$

where

$$\frac{\text{con2}}{\text{con4}} = 1 + \frac{C_{12}^2 T_3 + (C_{12} + C_{23})^2 T_2 + C_{23}^2 T_1}{\frac{\Sigma EI}{L^4} (T_1 T_2 + T_2 T_3 + T_3 T_1)} . \qquad 3.59$$

Now it is noted that

$$T_i = \frac{1}{EA_i} ,$$

$$C_{12} = \frac{r_2 - r_1}{L^2} ,$$

$$C_{23} = \frac{r_3 - r_2}{L^2}$$

and

$$C_{12} + C_{23} = \frac{r_3 - r_1}{L^2}$$

Substitution of the above relationships into equation 3.59 and simplifying yields

$$\frac{\text{con2}}{\text{con4}} = 1 + ((r_2^2-2r_2r_1+r_1^2)(A_1A_2)+(r_3^2-2r_3r_1+r_1^2)(A_3A_1)$$
$$+(r_3^2-2r_3r_2+r_2^2)(A_3A_2))\ ((\Sigma I)(A_1+A_2+A_3))^{-1}\ . \qquad 3.60$$

From the definition of the centroid it is noted that

$$r_1A_1 + r_2A_2 + r_3A_3 = 0\ . \qquad 3.61$$

If equation 3.61 is squared and the cross product terms are solved for, the resulting equation is

$$2r_2r_1A_2A_1 + 2r_3r_2A_3A_2 + 2r_1r_3A_1A_3 = -r_1^2A_1^2-r_2^2A_2^2-r_3^2A_3^2\ . \qquad 3.62$$

Substitution of equation 3.62 into equation 3.60 yields after some manipulation

$$\frac{\text{con2}}{\text{con4}} = 1 + \frac{A_1r_1^2 + A_2r_2^2 + A_3r_3^2}{\Sigma I}\ .$$

Equation 3.58 then becomes

$$\omega^2 = \frac{(n\pi)^4E(\Sigma I + A_1r_1^2 + A_2r_2^2 + A_3r_3^2)}{\Sigma\rho AL^4}. \qquad 3.63$$

For a solid beam such as that shown in Figure 3.3 (b), it is evident that

$$\rho A_s = \Sigma\rho A$$

and

$$I_s = \Sigma I + A_1r_1^2 + A_2r_2^2 + A_3r_3^2\ .$$

Equation 3.63 then becomes

$$\omega^2 = \frac{(n\pi)^4 EI_s}{\rho A_s L^4} \quad . \qquad 3.64$$

Equation 3.64 represents the eigenvalue equation for a solid Euler beam with simple supports and indicates that the solution will limit properly for very stiff connectors.

The solution for any given problem is simply a matter of calculating the problem constants and then using equation 3.57 for simply supported beams or forming a boundary condition matrix for alternate end conditions.

As a particular example, equation 3.57 was used to show the variation in the value of ω^2 for a system of three equal layers but with different interlayer connection properties. Figure 3.4 shows a plot of $\frac{\omega^2}{\omega_3^2}$ versus $\frac{K_{23}}{K_{12}}$ for various values of K_{12}. It is noted that

ω^2 = the first mode eigenvalue for the system

and

ω_3^2 = the first mode eigenvalue for a system with equal interlayer connection properties of $K = K_{12}$.

From this figure it is seen that for high and low values of K_{12}, a factor of two, difference between the interlayer properties is of little consequence. For intermediate values the error can be approximately 16%.

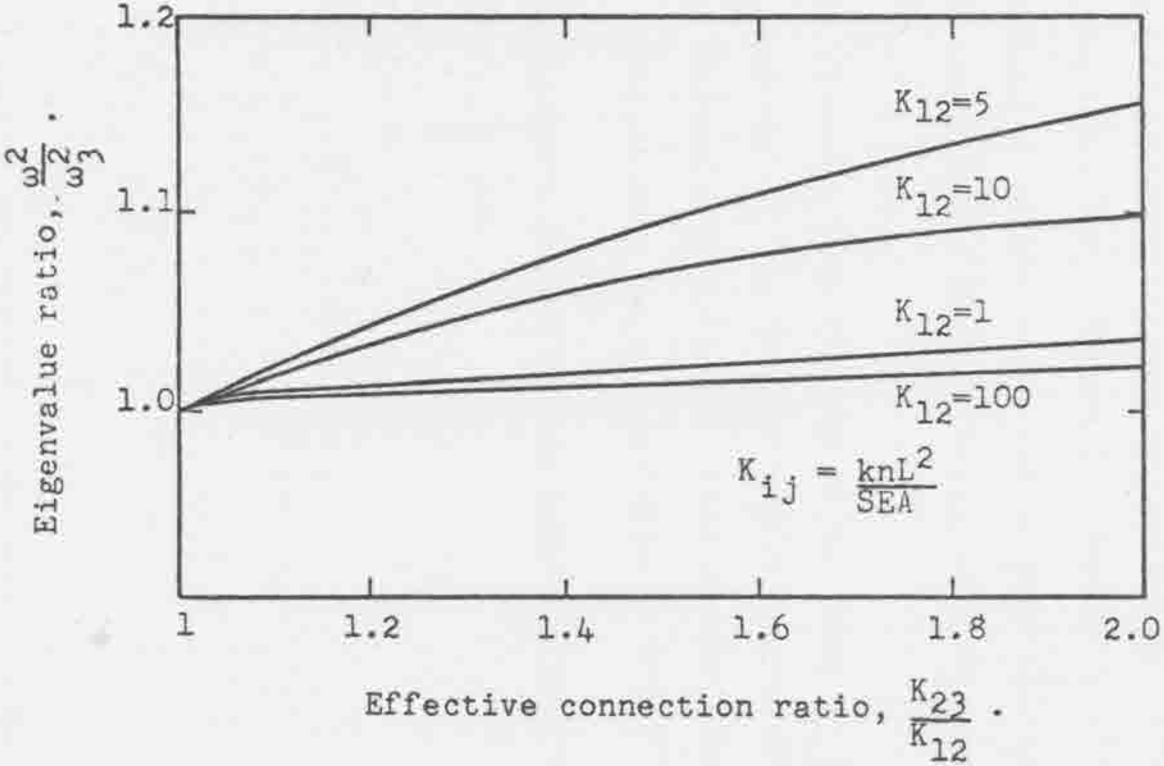


FIGURE 3.4 EFFECT OF UNEQUAL CONNECTOR MODULI ON A THREE EQUAL LAYER SYSTEM

3.4 N Layer System

The theory is here generalized to allow for a system of n layers with one axis of symmetry. As before, layers with different material properties may be treated by using the transformed cross-section. Figure 3.5 depicts a five layered system which serves as an aid in generalizing to n layers. From this figure the following nomenclature is noted:

$\bar{h}$ = the distance from the top of the beam to the centroid of the transformed cross-section (in.).

r_i = the distance from the centroid of the transformed cross-section to the centroid of the i^{th} layer.

Applying Newton's second law to the free body diagram in Figure 3.5 (c) yields

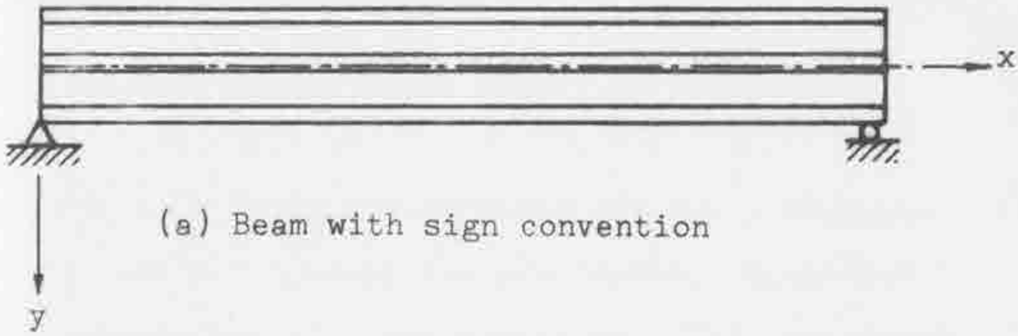
$$\sum_1^n F_{i,x} = 0 \tag{3.65}$$

$$V_{,x} = \sum_1^n \rho_i A_i Y_{i,2t} \tag{3.66}$$

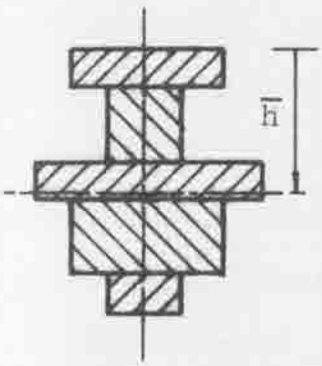
$$V = \sum_1^n M_{i,x} + \sum_1^n r_i F_{i,x} \tag{3.67}$$

As before all layers are assumed to deflect the same amount and have the same curvature, therefore

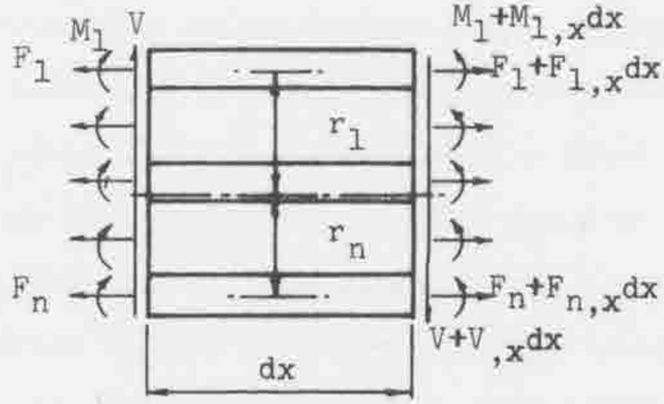
$$Y_{i,2t} = Y_{,2t}$$



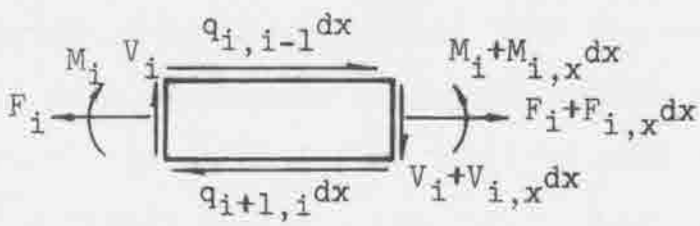
(a) Beam with sign convention



(b) Cross-section



(c) Beam element



(d) i^{th} layer element

FIGURE 3.5 N LAYER SYSTEM

and

$$M_i = -EI_i Y_{,2x} \; . \qquad 3.68$$

Equations 3.66 and 3.67 may now be combined to yield

$$\sum_1^n EI_i Y_{,4x} - \sum_1^n r_i F_{i,2x} + \sum_1^n \rho_i A_i Y_{,2t} = 0 \; . \qquad 3.69$$

For an n layer system there are n + 1 unknowns (nF's and Y), and n + 1 equations are needed. Equations 3.65 and 3.69 provide two of these equations. The additional n - 1 equations must come from slip relationships. For systems consisting of two or three layers these relationships can be obtained without recourse to the movements of the inner layers as has been done in previous sections. For four or more layers, however, this is not the case.

The term of interest is the relative slip displacement between adjacent layers. It is assumed that each layer slips as a block. As before, the slip permitted by a connector is directly proportional to the load transmitted by the connector. In a completely analogous manner the slip of a layer is directly proportional to the load transmitted to the layer. From Figure 3.5 (d), the force transmitted to the i^{th} layer in some dx length is $F_{i,x}dx$. For continuous shear connection

$$S_i = \frac{F_{i,x}}{(Keq)_i} \qquad 3.70$$

where

S_i = slip of the i^{th} layer in inches

and

$(Keq)_i$ = equivalent stiffness per unit length in lb/in/in.

Similarly

$$S_{i+1} = \frac{F_{i,x}}{(Keq)_{i+1}}$$

The relative slip is then

$$\Delta S_{i+1,i} = S_{i+1} - S_i$$

$$= \frac{F_{i+1,x}}{(Keq)_{i+1}} - \frac{F_{i,x}}{(Keq)_i} \, . \qquad 3.71$$

The displacement difference between the i + 1 and i layers may also be found by integrating the strains as

$$\Delta S_{i+1,i} = \int_o^x \epsilon_{i+1}^u dx - \int_o^x \epsilon_i^L dx \qquad 3.72$$

where

ϵ_{i+1}^u = the strain in the i + 1 layer evaluated at its upper boundary

and

ϵ_i^L = the strain in the i layer evaluated at its lower boundary.

From an inspection of Figure 3.5 (d) for tension taken as positive, equation 3.72 may be written as

$$\Delta S_{i+1,i} = \int_{o}^{x} \left\{ \left(\frac{F_{i+1}}{EA^{*}_{i+1}} - \frac{M_{i+1}}{EI_{i+1}} \frac{h_{i+1}}{2} \right) - \left(\frac{F_i}{EA^{*}_{i}} + \frac{M_i}{EI_i} \frac{h_i}{2} \right) \right\} dx \ . \qquad 3.73$$

Equations 3.71 and 3.73 are now combined and the substitution indicated by equation 3.68 is made to give

$$\frac{F_{i+1,x}}{(Keq)_{i+1}} - \frac{F_{i,x}}{(Keq)_i} = \int_{o}^{x} \left\{ \frac{F_{i+1}}{EA^{*}_{i+1}} - \frac{F_i}{EA^{*}_{i}} + \frac{1}{2}(h_{i+1} + h_i)Y_{,2x} \right\} dx \ . \qquad 3.74$$

Finally, equation 3.74 is differentiated to change the form to a differential equation. This yields

$$\frac{F_{i+1,2x}}{(Keq)_{i+1}} - \frac{F_i}{(Keq)_i} = \frac{F_{i+1}}{EA^{*}_{i+1}} - \frac{F_i}{EA^{*}_{i}} + C_{i+1,i}\, Y_{,2x} \qquad 3.75$$

where

$$C_{i+1,i} = \frac{1}{2}(h_{i+1} + h_i) \ .$$

Equation 3.75 provides the needed n - 1 slip relationships since in an n layered system there are n - 1 sets of adjacent layers.

Equations 3.65, 3.69, and 3.75 provide a system of n + 1 equations in n + 1 unknowns and represent the governing set of equations for an n layered system. The horizontal equilibrium equation may be used immediately to eliminate one of the F_i's, say F_n. This gives

$$\sum_1^n EI_i Y_{,4x} + \sum_1^{n-1}(r_n - r_i)F_{i,2x} + \sum_1^n \rho A Y_{,2t} = 0 \qquad 3.76$$

and

$$\frac{F_{2,2x}}{(Keq)_2} - \frac{F_{1,2x}}{(Keq)_1} = \frac{F_2}{EA_2^*} - \frac{F_1}{EA_1^*} + C_{21}Y_{,2x} \, ,$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{-\sum_1^{n-1} F_{i,2x}}{(Keq)_n} - \frac{F_{n-1,2x}}{(Keq)_{n-1}} = \frac{-\sum_1^{n-1} F_i}{EA_n^*} - \frac{F_{n-1}}{EA_{n-1}^*} + C_{n,n-1}Y_{,2x} \, . \qquad 3.77$$

Equations 3.76 and 3.77 provide a system of n equations in n unknowns. From an inspection of these equations it is seen that a total of 2(n + 1) boundary conditions are necessary for their solution. These boundary conditions may be found in a manner similar to that shown for the three layer problem. From an inspection of Figure 3.5 (c), the end moment may be written as

$$M_T = \sum_1^n M_i - \sum_{i=1}^n \left(\sum_{j=i+1}^n h_j + \frac{h_i}{2} \right) F_i \, .$$

Which may be rewritten as

$$M_T = -\sum_1^n EI_i Y_{,2x} - \sum_{i=1}^{n-1} \left(\frac{h_n}{2} + \sum_{j=i+1}^{n-1} h_j + \frac{h_i}{2} \right) F_i \, . \qquad 3.78$$

Similarly

$$V_T = \sum_1^n EI_i Y_{,3x} - \sum_{i=1}^{n-1} \left(\frac{h_n}{2} + \sum_{j=i+1}^{n-1} h_j + \frac{h_i}{2}\right) F_{i,x} \,. \qquad 3.79$$

Now following the reasoning used in Chapter 2, the boundary conditions may be written as

$$\begin{aligned} Y &= 0 \,, \\ Y_{,2x} &= 0 \,, \\ F_1 &= 0 \,, \\ &\;\;\cdot \\ &\;\;\cdot \\ F_{n-1} &= 0 \,, \end{aligned} \qquad 3.80$$

at a pinned end,

$$\begin{aligned} Y &= 0 \,, \\ Y_{,x} &= 0 \,, \\ F_{1,x} &= 0 \,, \\ &\;\;\cdot \\ &\;\;\cdot \\ F_{n-1,x} &= 0 \,, \end{aligned} \qquad 3.81$$

at a fixed end and

$$\begin{aligned} Y_{,2x} &= 0 \,, \\ F_1 &= 0 \,, \\ &\;\;\cdot \\ &\;\;\cdot \\ F_{n-1} &= 0 \,, \\ V_T &= 0 \,, \end{aligned} \qquad 3.82$$

at a free end.

Equations 3.80 through 3.82 indicate that at any end there are n + 1 boundary conditions. Thus, the necessary total of 2 (n + 1) boundary conditions are available.

The solution now proceeds as in previous sections. Equations 3.76 and 3.77 form a set of n, coupled, linear partial differential equations with constant coefficients. Hence, simple harmonic motion may be assumed such that

$$Y = Y(x)\cos(\omega t + \varphi)$$

and

$$F_i = F_i(x)\cos(\omega t + \varphi)\ .$$

If the above assumptions are made the equations may be expressed in linear operator form as

$$[L]\begin{Bmatrix} Y \\ F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_{n-1} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \qquad 3.83$$

where [L] is as shown in Figure 3.6 for

$$D^n = \frac{d^n}{dx^n}\ .$$

As before, the linear operator which provides the characteristic equation for Y or F_i may be found by setting the determinant of the operator matrix equal to zero.

$$L = 0 \qquad 3.84$$

$$[L] = \begin{bmatrix} \sum_{1}^{n} EI_i D^4 - \Sigma\rho_1 A_1 \omega^2 & (r_n - r_1)D^2 & (r_n - r_2)D^2 & \cdots & (r_n - r_{n-1})D^2 \\ C_{21}D^2 & \frac{1}{(Keq)_1} D^2 - \frac{1}{EA_1^*} & -\frac{1}{(Keq)_2} D^2 + \frac{1}{EA_2^*} & \cdots & 0 \\ C_{32}D^2 & 0 & \frac{1}{(Keq)_2} D^2 - \frac{1}{EA_2^*} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{n,n-1}D^2 & \frac{1}{(Keq)_n} D^2 - \frac{1}{EA_n^*} & \frac{1}{(Keq)_n} D^2 - \frac{1}{EA_n^*} & \cdots & \left(\frac{1}{(Keq)_n} + \frac{1}{(Keq)_{n-1}}\right) D^2 - \frac{1}{EA_n^*} - \frac{1}{EA_{n-1}^*} \end{bmatrix}$$

FIGURE 3.6 MATRIX OPERATOR FOR AN N LAYER SYSTEM

In general this operator will be of the form

$$D^{2(n+1)} + aD^{2n} + bD^{2(n-1)} + \ldots + u \qquad 3.85$$

where

n = the number of layers.

It is readily seen that once the equations are in the form given by equation 3.83, the problem can be solved in a manner similar to that of a two layer system. The algebraic complexity will, however, increase greatly.

Some discussion on the order of the operator as given by 3.85 is necessary. From 3.85 it is seen that in general the order of the operator depends upon the number of layers in the system. This contradicts the results of Pleshkov (23) and Rassam (24). Although their work concerned static bending and column buckling respectively, their results shown in Chapter 1 indicate that the operator should be independent of the number of layers in the system. In arriving at their results, both authors have used geometric constraints to eliminate some of the unknowns in a given problem. For the general case these constraints may be written in the nomenclature of this work as

$$\frac{\Delta S_{ij,x}}{\Delta S_{kL,x}} = \frac{C_{ij}}{C_{kL}} \quad . \qquad 3.86$$

The use of the above constraints allows for $n - 1$ of the F_i's to be solved in terms of the remaining F_i. This leaves only two dependent variables and following the procedure given for a two layer analysis, the operator reduces to the same order for all systems.

The constraints given by equation 3.86 are not true in general. They represent an approximation used for the sake of expediency, and their use can be shown to lead to erroneous results for certain cases. To see this a four layer system as shown in Figure 3.7 (a) is analyzed.

Consider first the results obtained by Pleshkov. His equation for static bending was given in Chapter 1 as

$$E\Sigma i_k \frac{d^4Y}{dx^4} - \frac{4G}{E\Omega}\left(EI_s \frac{d^2Y}{dx^2} + M\right) = - \frac{d^2M}{dx^2} .$$

From statics for a beam under a distributed load it is noted that

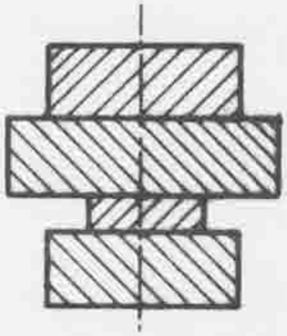
$$\frac{d^2M}{dx^2} = -q .$$

With the above substitution, differentiation twice yields

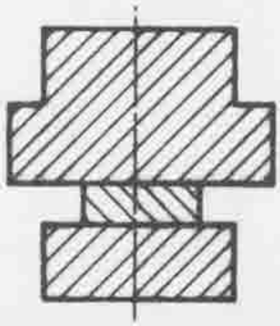
$$E\Sigma i_k \frac{d^6Y}{dx^6} - \frac{4G}{E\Omega}\left(EI_s \frac{d^4Y}{dx^4} - q\right) - \frac{d^2q}{dx^2} = 0 . \qquad 3.87$$

It is now assumed that the beam is not statically loaded but is rather undergoing simple harmonic motion such that

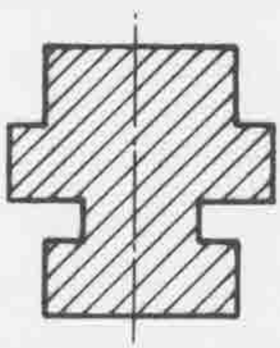
$$q = \Sigma\rho A\omega^2 Y .$$



(a) General four layer cross-section



(b) Actual reduced cross-section for G_1 very large



(c) Reduced cross-section for G_1 very large using Pleshkov assumption

FIGURE 3.7 FOUR LAYER CROSS SECTIONS WITH REDUCTION

Equation 3.87 may then be rewritten as

$$\frac{d^6Y}{dx^6} - \left(\frac{4G}{E\Omega}\frac{I_s}{\Sigma i_k}\right)\frac{d^4Y}{dx^4} - \left(\frac{\Sigma\rho A\omega^2}{E\Sigma i_k}\right)\frac{d^2Y}{dx^2} + \left(\frac{4G}{E\Omega}\frac{\Sigma\rho A\omega^2}{E\Sigma i_k}\right)Y = 0 \ . \qquad 3.88$$

Equation 3.88 has the same functional form as a two layer system with the only difference being in the form of the constants. From Chapter 1 it is noted that for a four layer system

$$G = \frac{G_1 + G_2 + G_3}{3} \qquad 3.89$$

Suppose that G_1 gets very large. The problem should reduce to a three layer system as shown in Figure 3.7 (b). However, from an inspection of equation 3.89 it is seen that as G_1 gets very large, G also gets very large, and as a result, equation 3.88 reduces to

$$\frac{d^4Y}{dx^4} - \frac{\Sigma\rho A\omega^2}{EI_s}Y = 0 \ . \qquad 3.90$$

Equation 3.90 is the governing equation for the free vibrations of a solid beam as shown in Figure 3.7 (c).

The fact that use of equation 3.86 leads to incorrect results for certain limiting cases shows that it provides at best an averaging process of the connector properties through the depth of the beam. This indicates that the geometric constraints are not valid if there are large differences in connector properties between layers.

Now consider the case where the interlayer connections are all the same. Equations 3.71 and 3.72 provide two expressions for relative slip. Using these expressions and the constraints provided by equation 3.86, the following relationships may be written after some manipulation

$$(F_{2,2x} - F_{1,2x})C_{32} = (F_{3,2x} - F_{2,2x})C_{21} \qquad 3.91$$

and

$$\left(\frac{F_{2,2x}}{A_2^*} - \frac{F_{1,2x}}{A_1^*}\right)C_{32} = \left(\frac{F_{3,2x}}{A_3^*} - \frac{F_{2,2x}}{A_2^*}\right)C_{21} . \qquad 3.92$$

The two relationships given by equations 3.91 and 3.92 should be the same. Note that if A_1^* is changed such that C_{21} remains the same (i.e., an increase or decrease in its base), then equation 3.92 will change while 3.91 will not. This indicates that the geometric constraints given by equation 3.86 are approximate even for the case of equal connector properties. If, however, a further restriction is placed on the problem such that the areas are equal, then equations 3.91 and 3.92 are equivalent. For this special case, equation 3.86 is exact and always valid.

Consider the case of four equal layers. Using the constraints given by equation 3.86, the following relationships may be written:

$$\frac{F_2 - F_1}{F_3 - F_2} = 1 . \qquad 3.93$$

$$\frac{F_2 - F_1}{F_4 - F_3} = 1 \; . \qquad 3.94$$

Now using equations 3.93 and 3.94 coupled with horizontal equilibrium, F_2, F_3, and F_4 may be solved in terms of F_1. Carrying out the indicated steps yields

$$F_2 = \frac{F_1}{3} \; ,$$

$$F_3 = \frac{-F_1}{3}$$

and

$$F_4 = -F_1 \; .$$

If these values are substituted into equations 3.69 and 3.75, the resulting relationships are

$$4EIY^{IV} + \frac{10}{3} hF_1'' - 4\rho A\omega^2 Y = 0 \qquad 3.95$$

and

$$hY'' - \frac{2}{3}\frac{S}{kn}F_1'' - \frac{2}{3}\frac{F_1}{EA} = 0 \; . \qquad 3.96$$

Now proceeding as with the two layer case, the equation in Y can be obtained as

$$Y^{VI} - 16\frac{kn}{SEA}Y^{IV} - \frac{\rho A\omega^2}{EI}Y'' + \frac{kn}{SEA}\frac{\rho A\omega^2}{EI}Y = 0 \; . \qquad 3.97$$

It is seen that for very large k, equation 3.97 reduces to

$$Y^{IV} - \frac{\rho A_s}{EI_s}\omega^2 Y = 0 \; . \qquad 3.98$$

Which is the governing equation for the free vibrations of a solid beam of height 4h.

For very small k, the equation for non-trivial Y is

$$Y^{IV} - \frac{\rho A\omega^2 Y}{EI} = 0 \ . \qquad 3.99$$

Which is the equation for the free vibrations of a solid beam of height h.

The use of the constraints have thus reduced the complexity of the problem for the equal layer case. The constraints are not necessary, however. The general formulation given by equations 3.83 and 3.84 should give the same results. For the case of four equal layers

$$C_{i+1,i} = h \ ,$$

$$(Keq)_i = \frac{kn}{S} = \frac{1}{G} \ ,$$

$$\frac{1}{EA^*_i} = \frac{1}{EA} = T$$

and

$$(r_4 - r_i) = (4 - i)h$$

Now using equation 3.84

$$\left| L \right| = 0$$

where

$$[L] =$$

$$\begin{bmatrix} 4EID^4-4\rho A\omega^2 & 3hD^2 & 2hD^2 & hD^2 \\ hD^2 & GD^2-T & -GD^2+T & 0 \\ hD^2 & 0 & GD^2-T & -GD^2+T \\ hD^2 & GD^2-T & GD^2-T & 2GD^2-2T \end{bmatrix} .$$

After some algebraic manipulation, the resulting equation for Y is found as

$$(D^2 - \frac{T}{G})(D^2 - \frac{T}{G})(D^6 - \frac{T}{G}(1 + \frac{5h^2}{4EIT})D^4$$

$$- \frac{\rho A\omega^2 D^2}{EI} + \frac{T}{G}\,\frac{\rho A\omega^2}{EI})Y = 0 . \qquad 3.100$$

It is noted that the general operator is tenth order. The operator equations of the form

$$(D^2 - \frac{T}{G})Y = 0$$

have solutions of the form

$$Y = a\sinh(\sqrt{\frac{T}{G}}\,x) + b\cosh(\sqrt{\frac{T}{G}}\,x) .$$

These solutions can satisfy the boundary conditions only for a = b = 0. By noting that

$$\frac{T}{G} = \frac{kn}{SEA}$$

and

$$1 + \frac{5h^2}{4EIT} = 16$$

the final equation for Y may be written as

$$Y^{VI} - 16\frac{kn}{SEA}Y^{IV} - \frac{\rho A\omega^2 Y''}{EI} + \frac{kn}{SEA}\frac{\rho A\omega^2 Y}{EI} = 0\ . \qquad 3.101$$

The equivalence of equations 3.97 and 3.101 indicates that the more general formulation of the problem gives the same results as the case when the approximation used by Pleshkov and Rassam is valid.

3.5 A Numerical Approximation

The closed form solutions obtained in previous sections depend in a large part on the assumption that the section properties and both connector modulus and spacing remain constant along the length of the beam. Once these restrictions are lifted, the resulting equations are best solved numerically.

Approximation techniques abound in the literature and are quite varied. Each has its strong and weak points, and no method provides a panacea (see for example ref. 33). It is the purpose of this section to provide an example of a numerical method which shows that the equations developed in previous sections are amenable to numerical solution. To this end the method presented will be chosen from the standpoint of simplicity and ease of understanding, rather than such use criteria as computational speed and accuracy. Of the many methods available, perhaps the most straightforward is the finite difference technique. It is this method that will be developed here as an example.

When approaching the problem by the finite difference method, it must first be decided whether to use the equations in their coupled or uncoupled form. The coupled equations have the advantages of using lower order approximations for derivatives, and uncomplicated boundary conditions. They have the disadvantage of requiring much higher order matrices for the same number of mesh points, as when the equations are reduced to a single uncoupled equation (38). From the standpoint of simplicity it is best to use the uncoupled equation even though the higher order derivatives limit accuracy.

The method developed will be applicable to simply supported beams whose general equation is of the form

$$Y^{VI} - AY^{IV} - B\omega^2 Y'' + C\omega^2 Y = 0 . \qquad 3.102$$

This limitation means that the beam must consist of two layers, three layers with dual symmetry or n equal layers. Additionally the fact that ω^2 appears in equation 3.102 indicates that the changes in properties are not of such a drastic nature that the simple harmonic motion assumption is violated. This eliminates working in x,t space and its associated solution stability problems.

It is assumed that the systems of the form given by equation 3.102 have been nondimensionalized with respect to length such that they may be written in the form

$$\frac{d^6Y}{dz^6} - aL^2\frac{d^4Y}{dz^4} - bL^4\omega^2\frac{d^2Y}{dz^2} + cL^6\omega^2Y = 0 \ . \qquad 3.103$$

Now taking

$$h = \frac{1}{n}$$

where n is the number of equal sections into which the beam is divided, and multiplying equation 3.102 by h^6 gives

$$h^6\frac{d^6Y}{dz^6} - \frac{aL^2}{n^2}\, h^4\, \frac{d^4Y}{dz^4} - \frac{bL^4\omega^2}{n^4}\, h^2\, \frac{d^2Y}{dz^2} + \frac{cL^6\omega^2}{n^6}\, Y = 0 \ . \qquad 3.104$$

Making the substitution

$$\delta^n = h^n\, \frac{d^n}{dz^n}$$

and writing equation 3.104 for the i^{th} beam element gives

$$\delta^6Y_i - \frac{a_iL^2}{n^2}\delta^4Y_i - \frac{b_iL^4}{n^4}\delta^2Y_i + \frac{c_iL^6}{n^6}\omega^2Y_i = 0 \ . \qquad 3.105$$

The following central difference operators may be obtained from a Taylor's expansion about the point Y(z+nh): (see for example reference 32).

$$\delta^2Y_i = Y_{i+1} - 2Y_i + Y_{i+1} \ .$$

$$\delta^4 Y_i = Y_{i+2} - 4Y_{i+1} + 6Y_i - 4Y_{i-1} + Y_{i-2} .$$

$$\delta^6 Y_i = Y_{i+3} - 6Y_{i+2} + 15Y_{i+1} - 20Y_i + 15Y_{i-1} - 6Y_{i-2} + Y_{i-3} .$$

Substitution of the central difference operators into equation 3.105 and collecting like terms yields

$$Y_{i-3} - R_1^i Y_{i-2} + (R_2^i - R_3^i \omega^2) Y_{i-1} - (R_4^i - R_5^i \omega^2) Y_i$$

$$+ (R_2^i - R_3^i \omega^2) Y_{i+1} - R_1^i Y_{i+2} + Y_{i+3} = 0 \qquad 3.106$$

where

$$R_1^i = \frac{a_i L^2}{n^2} + 6 ,$$

$$R_2^i = \frac{4a_i L^2}{n^2} + 15 ,$$

$$R_3^i = \frac{b_i L^4}{n^4} ,$$

$$R_4^i = \frac{6a_i L^2}{n^2} + 20$$

and

$$R_5^i = \frac{2b_i L^4}{n^4} + \frac{c_i L^6}{n^2}$$

Equation 3.106 provides the recursive formula for use in writing the finite difference equations. Prior to this, the boundary conditions must be imposed. This implies that the values of all Y_i outside the beam length must be specified in terms of Y_i inside the length. To this end, the boundary conditions must be specified in terms Y only. The conditions at a pinned end are

$$Y = 0 ,$$
$$Y'' = 0$$

and

$$F = 0 .$$

In Section 2.7 these conditions were found to be equivalent to

$$Y = 0 ,$$
$$Y'' = 0$$

and

$$Y^{IV} = 0 . \qquad 3.107$$

Suppose the beam in question has been divided into n segments and numbered from Y_o to Y_n. From the first of conditions 3.107 it is seen that

$$Y_o = Y_n = 0 . \qquad 3.108$$

Since Y_o and Y_n are known, it is only necessary to write equations for Y_1 through Y_{n-1}. Thus, from equation 3.106, it is seen that values of Y_{-2}, Y_{-1}, Y_{n+1}, and Y_{n+2} are needed. Using the difference operator for the second derivative, the condition

$$Y''(o) = 0$$

implies

$$Y_{-1} - 2Y_o + Y_1 = 0$$

and from equation 3.108

$$Y_{-1} = -Y_1 \ . \qquad 3.109$$

The condition

$$Y^{IV}(o) = 0$$

implies

$$Y_{-2} - 4Y_{-1} + 6Y_o - 4Y_1 + Y_2 = 0$$

and from equations 3.108 and 3.109

$$Y_{-2} = -Y_2 \ . \qquad 3.110$$

In a similar manner it is found that

$$Y_{n+1} = -Y_{n-1}$$

and

$$Y_{n+2} = -Y_{n-2} \ . \qquad 3.111$$

Using these boundary conditions, the recursive formula may now be used to write a system of n - 1 simultaneous linear homogeneous algebraic equations of the form

$$[R] \left\{Y\right\} = 0 \qquad 3.112$$

where

$$\left\{Y\right\} = \begin{Bmatrix} Y_1 \\ Y_2 \\ \cdot \\ \cdot \\ Y_{n-1} \end{Bmatrix}$$

The terms of interest in the set of equations 3.112 are the values of ω^2. To solve for these values it is necessary to change the form of the equations. Equation set 3.112 may be written as

$$[G]\ Y + \omega^2 [H] \left\{Y\right\} = 0 \qquad 3.113$$

where

$$[G] + \omega^2 [H] = [R] \ .$$

For the finite difference equations in the form given by equation 3.113, there are a number of techniques for solving for the eigenvalues. These techniques generally involve some rootfinder. For a non-singular G matrix, equation 3.113 may now be placed in standard form for performing a sweep iteration procedure as follows:

$$[G] \left\{Y\right\} = -\omega^2 [H] \left\{Y\right\} \ .$$

$$[G]^{-1} [G] \left\{Y\right\} = -\omega^2 [G]^{-1} [H] \left\{Y\right\} \ .$$

$$\frac{1}{\omega^2} [U] \left\{Y\right\} = [D] \left\{Y\right\} \qquad 3.114$$

where

$$[U] = [G]^{-1}[G] = \text{a unitary matrix}$$

and

$$[D] = -[G]^{-1}[H] \ .$$

Now using equation 3.114, the n - 1 eigenvalues and eigenvectors may be swept out from highest to lowest. The first eigenvalues are numerically more accurate than the last. Since they are found in decreasing numerical order, the first value found using 3.114 corresponds to the highest value of $\frac{1}{\omega^2}$, or the lowest value of ω^2.

A computer program was written to solve the finite difference equations. This program uses a standard matrix package available in most computer libraries. Illustrations of the G and H matrices for n equal to seven are shown in Figure 3.8 and a copy of the computer program is shown in Appendix III.

As a check on the method, the special case of three equal layers was run. For n equal to sixteen, the following results were obtained:

First mode K = 1	$\Psi = 1.74$	$\Psi_c = 1.736$
First mode K = 1000	$\Psi = 8.86$	$\Psi_c = 8.92$
Second mode K = 1	$\Psi = 1.16$	$\Psi_c = 1.20$
Second mode K = 1000	$\Psi = 8.47$	$\Psi_c = 8.70$

where

Ψ_c = closed form solution.

$$[G] =$$

$$\begin{bmatrix} R_1^1-R_4^1 & R_2^1-1 & -R_1^1 & 1 & 0 & 0 & 0 \\ R_2^2-1 & -R_4^2 & R_2^2 & -R_1^2 & 1 & 0 & 0 \\ -R_1^3 & R_2^3 & -R_4^3 & R_2^3 & -R_1^3 & 1 & 0 \\ 1 & -R_1^4 & R_2^4 & -R_4^4 & R_2^4 & -R_1^4 & 1 \\ 0 & 1 & -R_1^5 & R_2^5 & -R_4^5 & R_2^5 & -R_1^5 \\ 0 & 0 & 1 & -R_1^6 & R_2^6 & -R_4^6 & R_2^6-1 \\ 0 & 0 & 0 & 1 & -R_1^7 & R_2^7-1 & R_1^7-R_4^7 \end{bmatrix}$$

$$[H] =$$

$$\begin{bmatrix} R_5^1 & R_3^1 & 0 & 0 & 0 & 0 & 0 \\ R_3^2 & R_5^2 & R_3^2 & 0 & 0 & 0 & 0 \\ 0 & R_3^3 & R_5^3 & R_3^3 & 0 & 0 & 0 \\ 0 & 0 & R_3^4 & R_5^4 & R_3^4 & 0 & 0 \\ 0 & 0 & 0 & R_3^5 & R_5^5 & R_3^5 & 0 \\ 0 & 0 & 0 & 0 & R_3^6 & R_5^6 & R_3^6 \\ 0 & 0 & 0 & 0 & 0 & R_3^7 & R_5^7 \end{bmatrix}$$

FIGURE 3.8 MATRICES FOR FINITE DIFFERENCE SOLUTION

For first mode calculations the finite difference error was for all practical purposes negligible. For second mode eigenvalues, the error was on the order of four percent. Thus, it is seen that the finite difference technique gives reasonably good results for the simply supported case considered even though higher order derivatives were used.

The use of the uncoupled equation is not without problems, however. This can be seen by looking at the conditions at a fixed end for the three equal layer case. In Section 2.7 these conditions were found to be

$$Y = 0 \, ,$$

$$Y' = 0$$

and

$$Y^{V} - 8KY''' = 0 \, .$$

The third of the above conditions contains the fifth derivative as well as the third derivative. For an error of order h^2, the fifth derivative requires a seven point expansion, where a third derivative requires only a five point expansion. Thus, this boundary condition presents only one equation in two boundary unknowns. A symmetric extension of the deflections with respect to the fixed end will satisfy the third boundary condition, but in such a manner that

$$Y''' = 0 \, .$$

This extension leads to completely erroneous results. Thus, it is seen that the higher order derivatives in the boundary conditions provide complications. These complications can be overcome by using combined forward, backward, and central difference operators, or special techniques as reported by Gary and Helgason (37).

3.6 The Case of Damping

The mathematical models developed in previous sections do not account for the presence of damping in the systems to which the models apply. This is somewhat paradoxical in that the slipped system should provide inherently better damping than the material damping of the members, yet the mechanism which generally accounts for the largest part of the damping (i.e., friction) has been ignored. Nevertheless, the fact that the mathematical formulation does not explicitly contain damping terms does not preclude the consideration of damping altogether. By making certain assumptions and simplifications, a damped system may be investigated. The assumptions that are made are motivated by practicality and essentially involve the fact that the amount of damping present, though greater than material damping, is still small.

In Chapter 1 it was pointed out that slip damping is governed by Coulomb friction. When the amount of damping is small, the analysis of equivalent viscous damping may be used. This method assumes that the damping is viscous, that is, it is represented by a force proportional to the velocity, but opposite in direction. Thus, the damping is represented by a linear viscous model even though the system is following a different physical law of damping.

Meirovitch (11), among others, points out that in general damping produces a coupling of the normal coordinates. However, in the case in which damping is light, it is possible to obtain an approximate solution by considering any coupling due to damping as a secondary effect.

Using these assumptions, the equation of motion for the i^{th} normal mode with no forcing function may be written in terms of the generalized coordinates as

$$\ddot{\eta}_i(t) + C\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = 0 \qquad 3.115$$

where

η = generalized coordinate,

C = damping constant

and

ω_i = the natural frequency of the i^{th} normal mode which is found using the results of the previous sections.

Equation 3.115 is now solved in a standard manner by assuming that

$$\eta = \sum_{1}^{2} a_j e^{r_j t} . \qquad 3.116$$

Using the assumed solution in equation 3.115 yields the values of r_j as

$$r_j = -\frac{C_i}{2} \pm \sqrt{\frac{C_i^2}{4} - \omega_i^2} \; . \qquad 3.117$$

The behavior of the damped system depends upon the numerical value of the radical of equation 3.117. It is standard to use as a reference quantity a value of C which reduces the radical to zero. This value is called critical damping and is designated C_c. The actual damping in the system can be specified in terms of critical damping by using a nondimensional ratio called the damping ratio. This is defined as

$$\begin{aligned} \zeta_i &= \frac{C}{C_c} \\ &= \frac{C}{2\omega_i} \; . \end{aligned}$$

With these definitions, equation 3.117 may be written as

$$r_j = (-\zeta \pm \sqrt{\zeta^2 - 1}\,)\omega_i \; . \qquad 3.118$$

For the cases of interest here, ζ is less than one. By a simple redesignation of constants, equation 3.116 may be rewritten as

$$\eta_i = \bar{A}_i e^{-\zeta\omega_i t} \sin\,(\sqrt{1 - \zeta^2}\,\omega_i t + \varphi_i) \qquad 3.119$$

where $\bar{A}_i$ and φ_i are constants and depend upon how motion is initiated.

Equation 3.119 provides the solution for the generalized coordinates. It provides an oscillatory motion with diminishing amplitude. The frequency of the damped oscillation is given by

$$\omega_d = \sqrt{1 - \zeta^2}\omega_i \ .$$

For damping ratios on the order of one tenth or less, ω_d is essentially the same as ω_i.

It is emphasized that the solution governed by equation 3.119 provides an approximate answer only. However, this answer will give reasonable results even though the damping mechanism is more complex than the simple viscous model assumed under the conditions stated.

CHAPTER 4

SELECTED TESTS

4.1 Introduction

To provide an examination of the validity of the proposed theory, some simple tests were performed. The tests were limited to simply supported beams consisting of three equal layers. It was attempted to span the range of natural frequencies by varying the connector modulus. Thus, the connector properties were chosen without regard to structural significance.

Past experience with static bending and column buckling indicates that a proper knowledge of connector modulus is of prime importance in the slip system. For this reason effects due to EI variation were kept to a minimum by constructing the layered beams from aluminum layers.

In this chapter the test equipment and procedures used are detailed. The results obtained during testing are presented and compared with the proposed theory. Some discussion is also devoted to the problem of damping.

4.2 Test Equipment

The beams used in the experiments were fabricated using three layers of 6061-T6 aluminum. Each layer had nominal dimensions of 1/4 in. x 4 in. x 72 in., and was drilled with a 23/64 in. diameter drill at 2 in. centers both along the beam and across the width as shown in Figure 4.1 (a). The layers were joined together by press fitting different types of connectors into the pre-drilled holes.

The support system consisted of two simple supports spaced 70 in. apart, and anchored to a concrete floor by means of heavy bolts (see Figure 4.1 (b)). From the figure it is noted that the specimen actually rested on two points at one end and one at the other. This was done so as to minimize the damping associated with the support system and torsional effects.

Vibration of the beam was sensed by a moving shadow cast on a silicon photoelectric cell. The photoelectric cell was mounted with a direct current light source and was supported underneath the test specimen.

A shadow was cast on the photocell by means of a shadow vein. The shadow vein consisted of a thin, rectangularly shaped, piece of metal which was fastened to the test specimen by adhesive tape.

The voltage output of the photocell is proportional to the area of the cell which is exposed to light. The cell used was rectangular, and since the shadow vein cast a rectangular shadow, the voltage output of the cell is a linear function of the displacement of the test specimen.

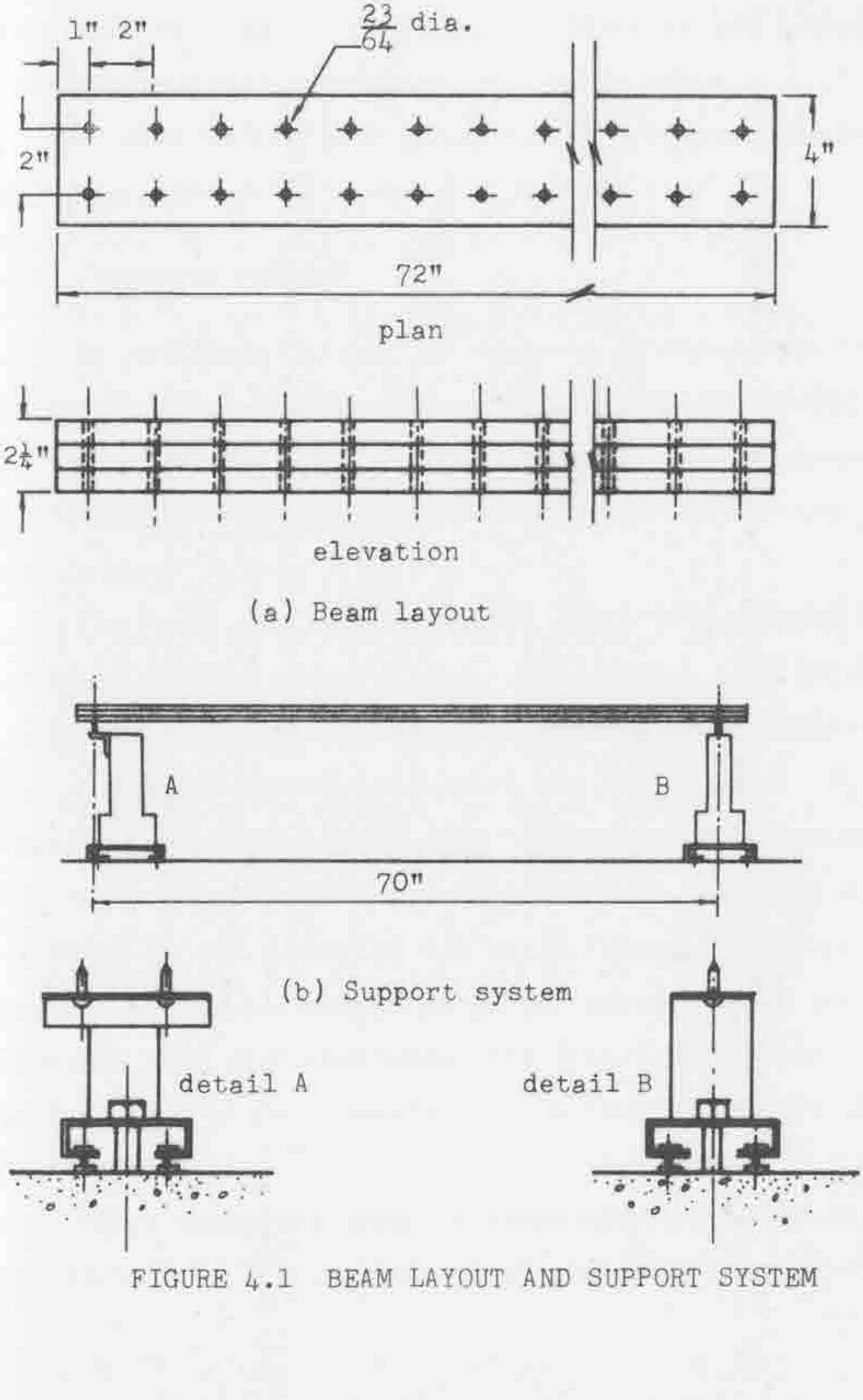


FIGURE 4.1 BEAM LAYOUT AND SUPPORT SYSTEM

The voltage was fed from the cell to a two channel Brush Mark 220 oscillograph through a balancing circuit. Thus, the vertical vibrational displacement was converted to time-displacement recordings on a strip chart.

Schematics and photographs of the vibrational testing system are shown in Figures 4.2 through 4.4.

4.3 Connector Modulus

As mentioned before, an accurate determination of the connector modulus is necessary if a reasonable prediction of system behavior is to be made. An accepted test procedure for connector modulus determination involves the use of a double shear joint.

The force was applied to the joint in a gradual manner using an Instron universal testing machine. The relative motion of the outer members of the joint with respect to the center member was measured using two dial guages. These measurements were averaged and the average value taken as the connector deformation. By taking the total force and dividing by the number of connector facings, a plot of connector deformation versus force per connector was made. The slope of this plot represents the connector modulus. A typical plot and a schematic of the test set up are shown in Figure 4.5.

Three different types of connectors were used in making the specimens. These were rubber, low density polyethylene,

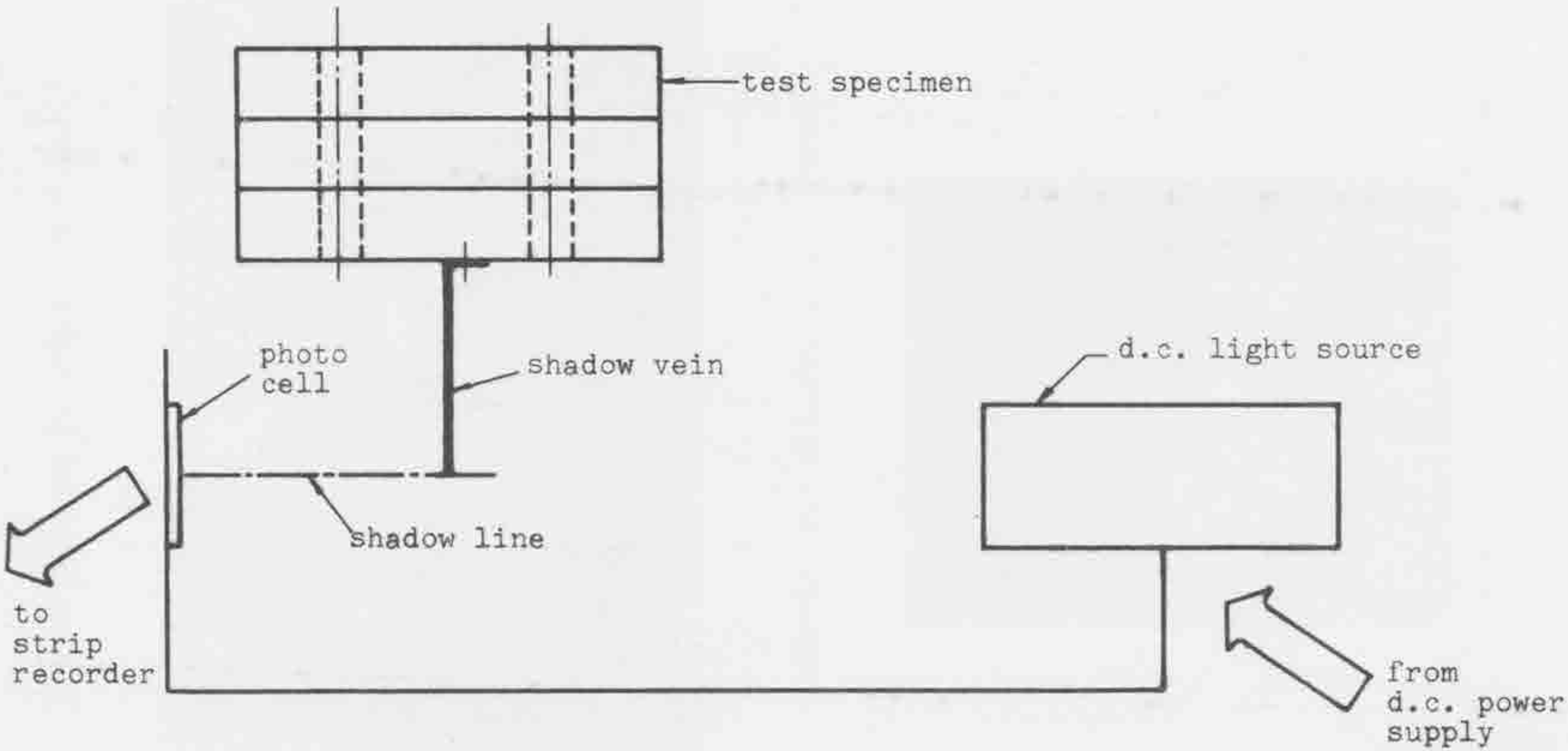


FIGURE 4.2 SCHEMATIC OF VIBRATION SENSING SYSTEM

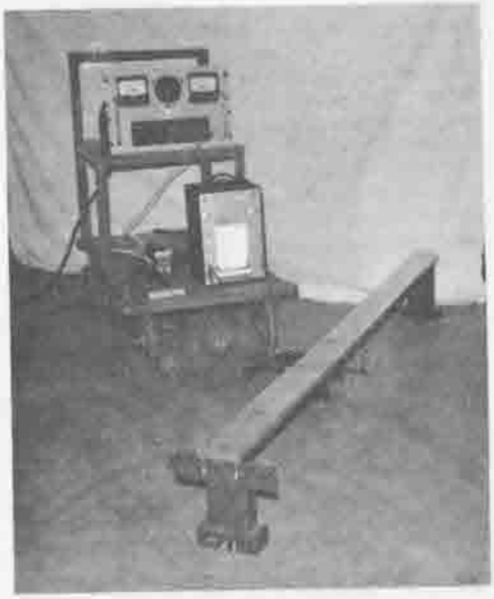


FIGURE 4.3 VIBRATION TEST SYSTEM

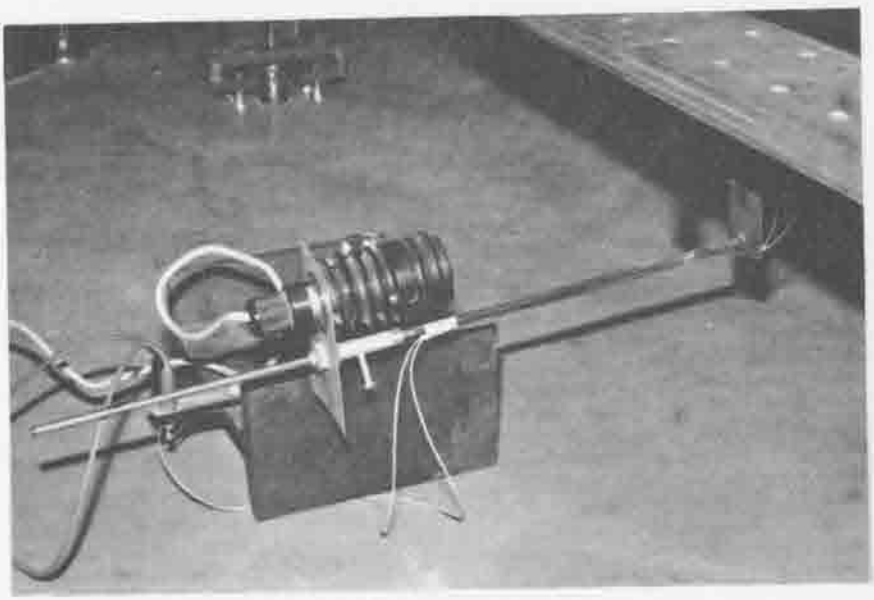
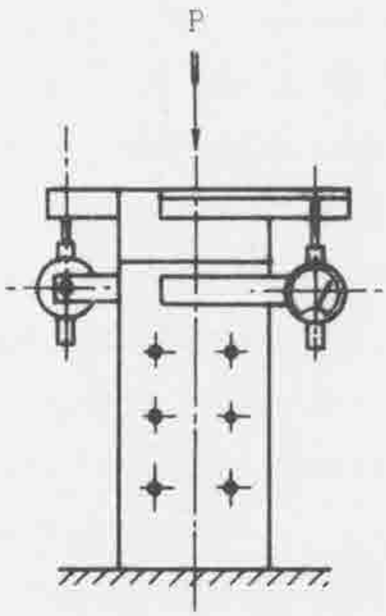
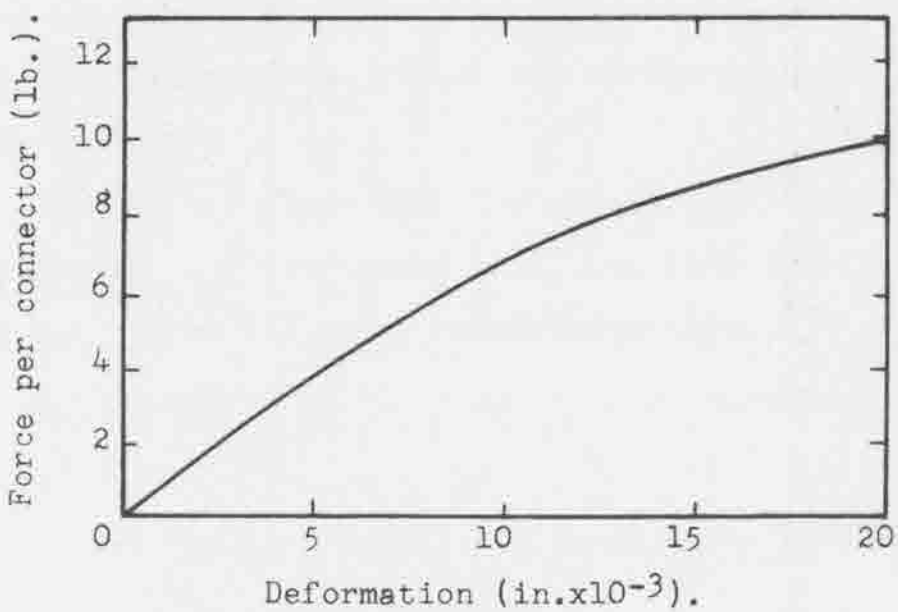


FIGURE 4.4 VIBRATION SENSING SYSTEM



(a) Test apparatus



(b) Force versus deformation for rubber connector

FIGURE 4.5 SLIP TEST ARRANGEMENT

and nylon. The connectors were cut from long sections of material. It was found that the polyethylene and nylon specimens had variations in diameter of some 18 thousandths of an inch. Since the properties of polymers change drastically when cold worked (i.e., pounding into the pre-drilled holes), it was felt that there could be considerable variation in slip properties between the test sample and the beam itself. For this reason an additional test was made for connector modulus.

The additional test was made on the total beam. The beam was placed on the supports with a concentrated load at midspan. The displacement at midspan was then measured. Using the load and deflection, the connector modulus could be determined with the aid of the theory developed by J. R. Goodman (1, 2).

Goodman has shown that for a beam with three equal layers under a concentrated load, the deflection of the beam is given by the following coupled equations:

$$Y = Y_s + \frac{8}{9}\frac{S}{kn}\frac{1}{h}F_L . \qquad 4.1$$

$$F_L = -\frac{C_2}{C_1}\frac{P}{\sqrt{C_1}}\frac{\sinh[\sqrt{C_1}(L-U)]\sinh\sqrt{C_1}X}{\sinh\sqrt{C_1}L} + \frac{C_2}{C_1}P(1-\frac{U}{L})x . \qquad 4.2$$

where

Y = deflection of the beam,

Y_s = deflection of the equivalent solid beam,

F_L = axial force in the left hand end of the beam,

$C_1 = \frac{9kn}{SEA}$,

$C_2 = \frac{knh}{3SEI}$,

P = load,

U = point of load application measured from the left hand end as a fraction of L

and

L = length of the beam.

For the case when U and x are equal to $\frac{L}{2}$, equations 4.1 and 4.2 may be combined and written in the form

$$\Delta Y = \frac{C_2 P}{2C_1} [\frac{L}{2} - \frac{1}{\sqrt{C_1}} \tanh(\sqrt{C_1} \frac{L}{2})][\frac{8}{9}\frac{S}{knh}] \qquad 4.3$$

where

$\Delta Y = Y - Y_s$.

Equation 4.3 can be used to find the value of k by assuming values of k and checking to see when the right hand side is equal to the known left hand side. This method was used for an additional determination of the connector modulus.

4.4 Test Procedure

In addition to the connector modulus tests previously described, some preliminary tests were made on the individual layers prior to construction of a test beam. These tests consisted of simply measuring, weighing, and vibrating the individual layers to determine values of certain physical parameters such as mass per unit length, moment of inertia, and modulus of elasticity.

A vibration technique was used for the modulus of elasticity measurement. For a simply supported beam vibrating in the first mode, the natural frequency is given by

$$\omega = \pi^2\sqrt{\frac{EI}{\rho AL^4}}$$

Solving for E yields

$$E = \left(\frac{\rho AL^4}{I}\right)\left(\frac{\omega}{\pi^2}\right)^2 . \qquad 4.4$$

For all other constants known and ω measured, a value of the modulus of elasticity can be determined from equation 4.4.

When a test specimen had been constructed, it was placed on the support system with the shadow vein attached. The electrical system was then balanced between the photocell and brush recorder, and the beam was ready to test. The specimen was set into vibration by depressing the center point until the recorder pen was just off-scale, and then quickly releasing the specimen to excite the first natural frequency.

To test repeatability of the results, each specimen was tested a total of six times, twice on each of the flat sides and twice with the beam reversed on the supports.

From each of the time-displacement graphs, data were collected for the computations which were performed. The data included the number of cycles in a given length of strip-chart and the amplitude of the first and some other cycle. These allowed for the determination of natural frequency and damping ratio.

The natural frequency was determined by multiplying the number of cycles occurring in some chart length by the speed of the chart divided by the chart length. That is

$$f_n = \text{number of cycles} \times \frac{\text{chart speed (mm/sec)}}{\text{chart length (mm)}}$$

where

f_n = the natural frequency in hertz.

The decrease in vibration amplitude gave a measure of the damping present. This was done by making use of a quantity known as the logarithmic decrement. The logarithmic decrement can be defined as

$$\delta = \frac{1}{n} \ln \left(\frac{\overline{A}_o}{\overline{A}_n}\right) \qquad 4.5$$

where

δ = logarithmic decrement,

n = number of cycles,

$\overline{A}_o$ = initial amplitude

and

$\overline{A}_n$ = amplitude after n cycles.

For the motion given by equation 3.119, it is noted that

$$\frac{1}{n}\ln\left(\frac{\bar{A}_o}{\bar{A}_n}\right) = \frac{1}{n}\ln e^{n\zeta\omega_n\tau}$$

$$= \zeta\omega_n\tau \ . \qquad 4.6$$

where

τ = period of damped oscillation

$$= \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \ . \qquad 4.7$$

If equations 4.5, 4.6, and 4.7 are combined for the case when ζ is small such that its square can be neglected with respect to one, the resulting equation is

$$\zeta = \frac{1}{2\pi n}\ln\left(\frac{\bar{A}_o}{\bar{A}_n}\right) \ . \qquad 4.8$$

Using two amplitudes taken from the decay curve and equation 4.8, the damping ratio for the system tested was determined.

4.5 Test Results

The results from the tests of the physical properties of the layers and the connectors are shown in Tables 4.1 and 4.2. It is noted that the length specified in Table 4.1 is the true length of the beam whereas the length used in vibration equations is the span length (i.e., 70 in.). The modulus of elasticity value specified is an average value taken from vibration of the individual layers.

TABLE 4.1 PHYSICAL PROPERTIES OF THE LAYERS

Layer	Length in.	Height in.	Area $in.^2$	Weight Grams	E $lb./in.^2 x10^{-6}$
1	72	.256	1.02	3282	10.5
2	72	.255	1.02	3270	10.6
3	72	.256	1.02	3280	10.5

TABLE 4.2 CONNECTOR PROPERTIES

Connector	n	S in.	k_L $lb./in.x10^{-3}$	k_h $lb./in.x10^{-3}$
Rubber	2	2	.525	.700
Polyethylene	2	2	2.75	3.80
Nylon	2	4	90.0	150.

In Table 4.2 it is seen that two values of the connector modulus appear. These are denoted as k_L and k_h which are low values and high values respectively. For the rubber connectors, the two values arose from the fact that the connector modulus was nonlinear. Instead of treating the problem in a nonlinear fashion, it was decided to take two linear values of connector modulus from the plot of deformation versus force applied. For the polyethylene and nylon connectors the connector modulus was linear. However, due to the problems mentioned in section 4.3, two different measurements were taken. For both polyethylene and nylon connectors the high value of the connector modulus came from using the beam test as given by equation 4.3 and the low value from direct measurement. The spread between the high and low measurements of connector modulus represents an uncertainty band in determining the true value of the connector modulus.

The test results for the layered systems are presented in Table 4.3. Using the average value of natural frequency, the results can be compared to the theoretical results in graphical form as shown in Figure 4.6. Typical strip-charts are shown in Figure 4.7. The chart speed used in the tests was 25 millimeters per second.

4.6 Discussion of Results

In general, the tests agreed quite favorably with the proposed theory as far as the ability to predict natural frequency is concerned. Figure 4.6 indicates that in each case the theoretical value is contained within the uncertainty span of connector modulus.

There are certain factors that could cause the system to deviate from an ideal three equal layer system. It is felt that the fact that the layers were drilled had small effect on the assumption of constant EI along the length. This leaves only the connector properties as a real source of deviation.

TABLE 4.3 LAYERED SYSTEM TEST RESULTS

Test	f cyc/sec	Ψ	ζ
R1	5.35	1.22	.046
R2	5.35	1.22	.050
R3	5.35	1.22	.048
R4	5.35	1.22	.049
R5	5.35	1.22	.047
R6	5.35	1.22	.049
$\overline{R}$	5.35	1.22	.048
P1	7.27	2.23	.079
P2	7.27	2.23	.076
P3	7.27	2.23	.079
P4	7.16	2.17	.075
P5	7.27	2.23	.074
P6	7.27	2.23	.072
$\overline{P}$	7.25	2.22	.076
N1	12.8	6.94	.0065
N2	12.8	6.94	.0074
N3	13.0	7.16	.0065
N4	12.8	6.94	.0092
N5	13.0	7.16	.0092
N6	13.0	7.16	.0069
$\overline{N}$	12.9	7.07	.0076

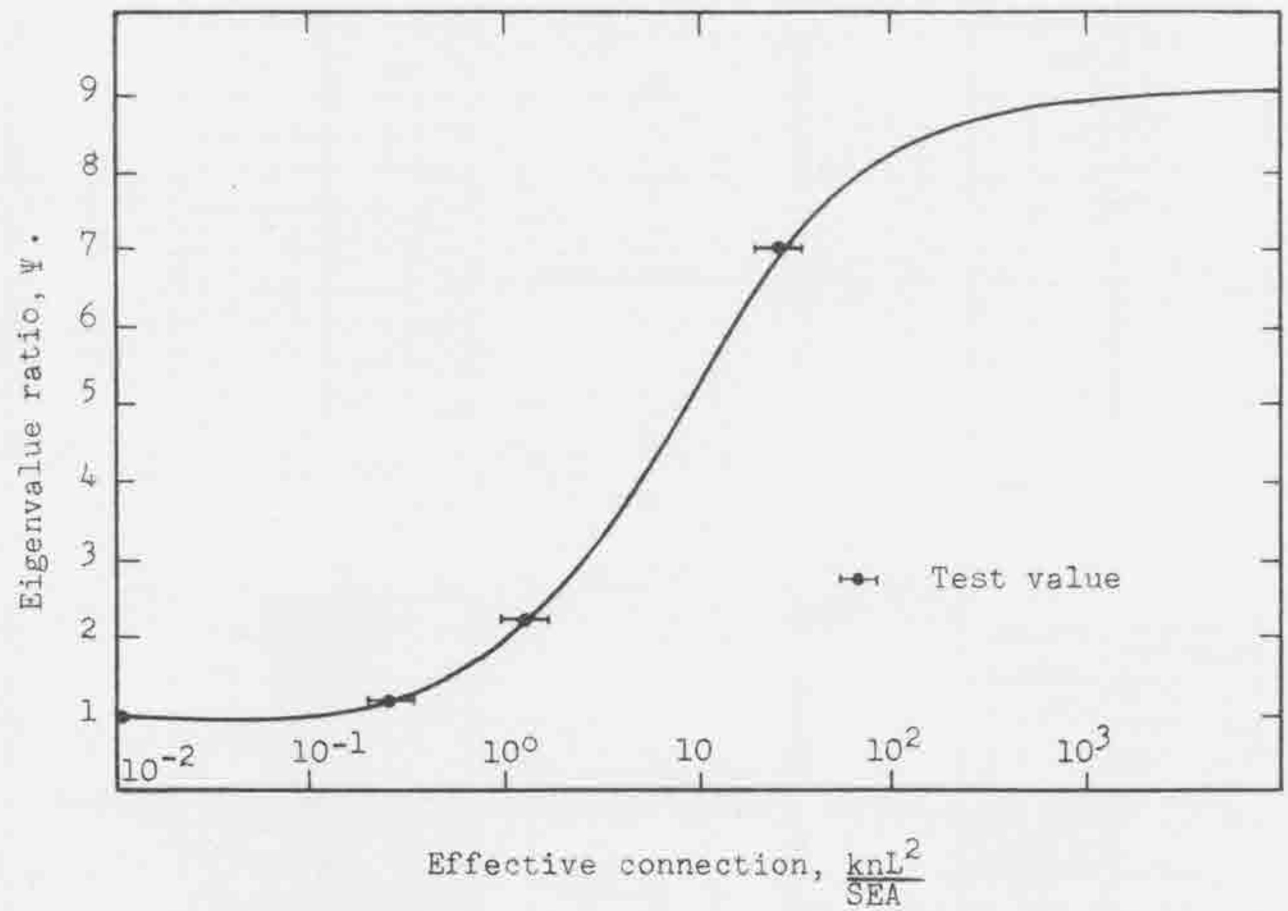
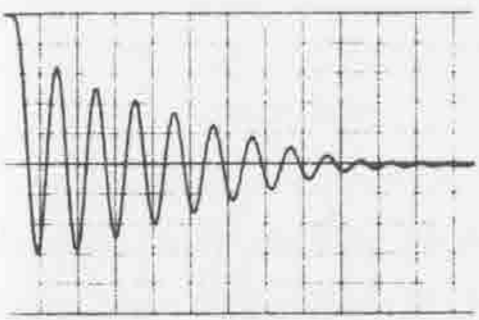
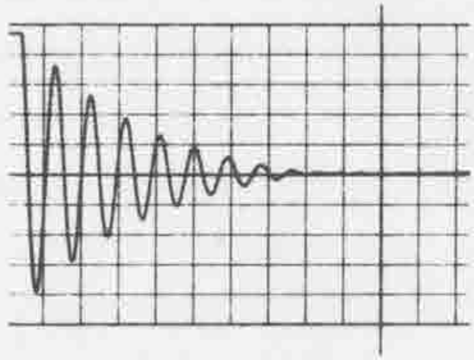


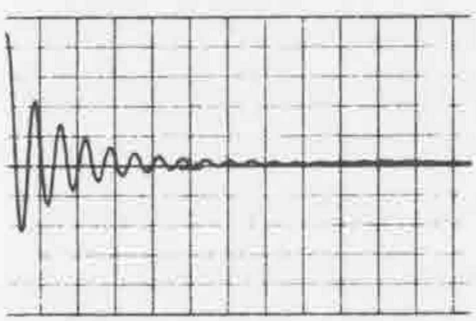
FIGURE 4.6 TEST RESULTS VERSUS THEORY



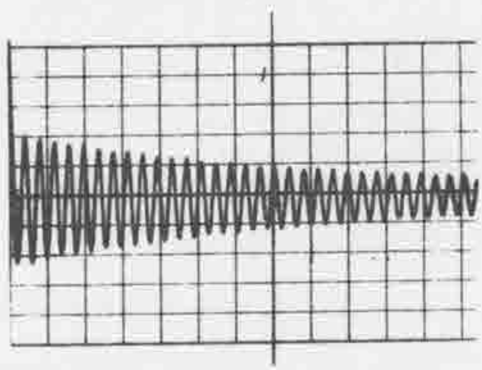
(a) Unconnected system



(b) Rubber connectors



(c) Polyethylene connectors



(d) Nylon connectors

FIGURE 4.7 TYPICAL STRIP CHARTS

There are two possible ways in which the connectors can produce deviations from the ideal system. These are that the interlayer connections differ between layers, and that the connector modulus is not constant along the length.

The first source should have almost no effect on the results of this test. This is because an average value was taken during the slip test as explained in Section 4.3. During this testing the deformation of one side was never more than twice that of the other. This indicates that the worst disparity between k_{12} and k_{23} would be less than a factor of two. The natural frequency can be calculated using the general three layer theory for k_{23} twice the value of k_{12}. It can also be calculated using the three equal layer equations with an average value of k. When the two results were compared, there was less than a 1% difference.

This leaves the fact that the connector modulus was not constant along the length of the beam as a primary source of uncertainty. The large disparity in the moduli as measured by direct slip testing and by beam theory indicate that the connector modulus was in fact not a constant. This is borne out by the fact that certain connectors had to be cold worked much more than others as described in section 4.3.

It is emphasized that it was attempted to attain a constant connector modulus even though a non-constant modulus resulted. Non-constant connector moduli are more representative of reality than constant ones. This is especially true of nailed wood structures. Even though it is possible to obtain a solution for non-constant moduli by numerical means, the assumption of a constant modulus gives quite reasonable results if one is willing to accept a range of uncertainty in the ability to quantify the connector modulus.

The damping ratios found during testing were all less than 10% of critical. These ratios are small, and assumptions made in arriving at the damped solution given by equation 3.119 are justified. Even though the damping mechanism is different than a linear viscous model, the viscous, small damping solution gives an adequate representation of the vibration decay envelope for the range of damping investigated. Thus, given a damping ratio for a beam system, the developed solution, given by equation 3.119, will provide a proper representation of the time-displacement behavior of the system.

Even though the damping was small, there was a significant increase in the damping associated with the slipped system as compared to material damping (i.e., material damping ratio for aluminum .000003). However, the damping ratios measured included the effects of support dissipation, connector damping, and air damping as well as the slip effect. If damping is to be optimized it would be necessary to separate these effects from one another. To do this would require extremely well-controlled tests. This is seen from the fact that even though it was attempted to minimize support damping and to make each test exactly the same, the nylon tests showed a variation in damping ratio.

CHAPTER 5

SUMMARY AND CONCLUSIONS

5.1 Summary

In this study, a theory was developed for the transverse vibrations of layered beams with the effects of interlayer slip included. Starting with the special case of a three equal layer system, the governing equations for beams of increasing complexity, up to an N layer system with one axis of symmetry, were presented. It was shown that in general the number of simultaneous equations necessary to describe the system behavior is dependent upon the number of layers in the system. Upon uncoupling it was shown that an N layer system yields a single differential equation of order $2(N + 1)$. Additionally it was shown that the order of the equation could be reduced using geometric constraints such that a system of N equal layers will yield a sixth order equation upon uncoupling.

The boundary conditions necessary to solve for the natural frequencies were presented. It was shown that the necessary number of boundary conditions such that the problem is well posed is available. The boundary conditions were checked by the use of the principle of minimum energy as a means of developing the equations of motion.

A solution technique was employed to solve the equations in closed form for various sets of the boundary conditions. The validity of the equations was tested by comparing their solutions at their upper and lower limits with well known results for Euler type beams. In all cases the limiting solutions were shown to give expected results, for example, very stiff connectors tend to limit to the equivalent solid beam.

Several non-dimensional graphs were generated to show the effects of variation in the physical and geometric properties of the layers and in the connector properties on the natural frequencies of the layered system. These graphs allow for the prediction of the system natural frequency for known values of certain problem constants.

The problem of damping was considered in a simplified sense. Motivated by the practicality of obtaining a solution, it was assumed that the highly complex damping mechanism associated with slip could be replaced with a linear viscous model and that the damping was small. These simplifications negated any coupling due to damping and allowed for solution of the decay envelope in a standard manner.

In addition to testing solutions at their limits, further verification of the theory was attempted through experimental means. Some simple tests were performed for the special case of a three equal layer system. The natural frequencies predicted by the theory were compared with the test results. The agreement between theory and test was generally good. The tests also showed that for the connectors considered, the linear viscous damping solution gave an adequate representation of the damped system behavior, and that the slipped system possesses good energy dissipation capability.

5.2 Conclusions

The theory presented in this study allows for the prediction of the vibrational behavior of layered beams with interlayer slip within the confines of the small damping assumption. The system behavior is largely dependent upon the strength of the interlayer connection. As the connection becomes stronger the slip effects tend to disappear and the system approaches a solid Euler beam. Additionally it appears feasible that where stiffness can be sacrificed, improved damping is available by increasing slip.

Although the theory developed is directly applicable to vibration problems, it is evident that the problems of static bending and buckling may also be considered. The general treatment presented here does not involve the use of the geometric approximation employed by Pleshkov (23) and Rassam (24). Thus, the exact nature of this approximation can be understood and limitations as to its applicability can be found. This appears to be an area worthy of further consideration.

During the course of this study, some questions arose that should be given further effort. In general, these questions concern the connector. As stated before, the connector modulus is of prime importance in predicting system behavior. Therefore, it is necessary to know its value as accurately as possible, or to know within what bounds it can be predicted. This indicates the need for experimental work such that a rational procedure is available for predicting the connector stiffness properties as it will occur in a "non-laboratory" situation.

The other area in which the connector properties are important is that of system damping. An accurate prediction of damping or a method of construction whereby damping can be optimized can only come about after a large experimental effort. Although previous efforts have shown that the hysteresis loops associated with wood systems do in fact reach a reproducible state and lend themselves to being replaced by their viscous equivalents, no method exists for an accurate prediction of what the system damping capacity will be. That is, for a given connector system, the ability to predict the damping ratio with any confidence in its accuracy is doubtful. This fact is borne out by tests run on the three layer system. The total system damping is dependent upon the damping of the connectors themselves as well as that due to interlayer friction. The maximum total damping was not obtained from the system where the slip was maximum but rather at some lesser value of system slip. Thus although the frictional damping tends to decrease since there is less slip, the total damping increases. Therefore, although it is reasonable to state that a system with interlayer slip has greater energy dissipation capabilities than does a system without slip, the quantification of the damping is not yet possible.

A study of the entire area of connector properties should provide a ground for fruitful efforts. As the knowledge of this area increases, so will the ability to predict system behavior.

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APPENDICES

APPENDIX I

LIST OF SYMBOLS

A_i = Area of the i^{th} layer

A_i^* = Transformed area of the i^{th} layer

$\overline{A}_i$ = Amplitude of vibration for the i^{th} normal mode

A_s = Area of a solid cross-section

a_i = Constant in polynomial equation

$\overline{a}_i$ = Constant in assumed solution

b_i = Constant in assumed solution

[BC] = Matrix of boundary conditions

C = Damping constant

C_c = Critical damping constant

c_i = Constant relating Y and F solutions

C_{ij} = Centroidal distance between i^{th} and j^{th} layers

Con = Constant in general three layer solution

D^n = Operator indicating the n^{th} derivative with respect to the space variable

$[D]$ = Matrix in numerical solution

d_i = Root of polynomial equation

Dis = Discriminant of third degree polynomial

E = Modulus of elasticity

F_i = Axial force in the i^{th} layer

f_n = Natural frequency in cycles per second

G = Average connector modulus used by Pleshkov

G_{ij} = Flexibility per unit length between i^{th} and j^{th} layers

$[G]$ = Matrix in numerical solution

$[H]$ = Matrix in numerical solution

h = Height in equal layer system

h_i = Height of the i^{th} layer

$\overline{h}$ = Height from top of beam to centroid of cross-section

I_i = Moment of inertia of the i^{th} layer

I_s = Moment of inertia of equivalent solid beam

$i()$ = Imaginary quantity ()

K = Effective connection

Keq_i = Equivalent connection for single shear test

k = Connector modulus

k_h = High estimate of connector modulus

k_L = Low estimate of connector modulus

$[L]$ = General matrix operator

L = Beam length

M_i = Bending moment in the i^{th} layer

M_T = Total moment on a cross-section

m = Mass per unit length of beam

N = Ratio of mass per unit length of beam to the summation of layer bending stiffnesses

n = An integer number

P = Point load

p_i = Root of reduced polynomial

Q = Load transmitted by a connector

q = Distributed load

q^* = Constant in determining Dis

q_{ij} = Load transmitted between the i^{th} and j^{th} layers

R_i^n = i^{th} recursive constant for n^{th} beam element

r_i = Distance from centroid of beam to centroid of i^{th} layer

r^* = Constant in determining Dis

S_i = Slip of i^{th} layer

s = Connector spacing

T_i = Axial stiffness per unit length for the i^{th} layer

T.C. = A transfer constant relating I_s and the summation of I

t = Time

V = Shear force on beam element

V_i = Shear force on the i^{th} layer element

V_T = Total shear

x = Space variable

Y_i = i^{th} beam normal mode

Y = Beam deflection used by J. R. Goodman

Y_s = Solid beam deflection used by J. R. Goodman

z = Dimensionless space variable

α = Root to sixth order polynomial

β = Root to sixth order polynomial

γ = Root to sixth order polynomial

ΔS_{ij} = Interlayer slip between i^{th} and j^{th} layers

ΔY = Additional deflection due to slip used by Goodman

δ = Logarithmic decrement

δ^n = n^{th} central difference operator

ϵ_i^L = Strain in the i^{th} layer at the lower edge

ϵ_i^u = Strain in the i^{th} layer at the upper edge

ζ = Damping ratio

η = Generalized coordinate

λ = Lagrange multiplier

ξ = Ratio of two layer eigenvalue to solid beam eigenvalue

ρ_i = Mass per unit volume of i^{th} layer

σ = Stress level

τ = Period of damped vibration

φ = Phase angle

ψ = Ratio of three layer eigenvalue to single layer eigenvalue

Ψ_c = Closed form eigenvalue ratio

ω = Natural frequency in radians per second

ω^2 = Eigenvalue

APPENDIX II

ALTERNATE EQUATION DEVELOPMENT

In this section the equations of motion for the three equal layer beam are developed from a variational standpoint. Detailed discussions of such terms as Lagrangian multiplier, Hamilton's principle, etc. may be found in references 11, 12, 19, and 34.

Hamilton's principle reduces a problem in dynamics to the investigation of a scalar integral. The condition which renders the value of the integral stationary leads to all the associated equations of motion. A mathematical statement of this principle is

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \qquad \text{II-1}$$

where

T = kinetic energy

and

V = potential or strain energy.

For the case of three equal layers, the kinetic energy is simply

$$T(t) = \frac{1}{2} \int_{o}^{L} 3\rho A\left[\frac{\partial Y}{\partial t}\right]^2 dx \;. \qquad \text{II-2}$$

The strain energy will come from three sources. These will be called bending, axial and slip which is a measure of the energy stored in the connectors due to fact that the layers slip relative to one another.

For bending strain energy

$$V_b(t) = \frac{1}{2} \int_o^L \sum_1^3 \text{(moment)(curvature) } dx$$

$$= \frac{1}{2} \int_o^L 3EI \left[\frac{\partial^2 Y}{\partial x^2}\right]^2 dx \, . \qquad \text{II-3}$$

For axial strain energy

$$V_a(t) = \frac{1}{2} \int_o^L \Sigma F_i \Delta x_i$$

$$= \frac{1}{2} \int_o^L \frac{2F^2}{AE} \, dx \, . \qquad \text{II-4}$$

For slip strain energy

$$V_s(t) = \frac{1}{2} \int_o^L \Sigma \text{ (force causing slip)(slip distance)}$$

$$= \frac{1}{2} \int_o^L [q_c \Delta S_{12} + q_c \Delta S_{23}] \, dx \, .$$

Noting from Chapter 2 that

$$\Delta S_{ij} = q_c \frac{S}{kn} \, ,$$

and

$$q_c = \frac{\partial F}{\partial x}$$

yields

$$V_s(t) = \frac{1}{2} \int_o^L \frac{2S}{kn}\left(\frac{\partial^2 F}{\partial x^2}\right)^2 dx \ . \qquad \text{II-5}$$

Substitution into equation II-1 gives

$$\delta \int_{t_1}^{t_2} \left\{ \int_o^L \left[\frac{3\rho A}{2}\left(\frac{\partial Y}{\partial t}\right)^2 - \frac{3EI}{2}\left(\frac{\partial^2 Y}{\partial x^2}\right)^2 - \frac{1}{AE} F^2 - \frac{S}{kn}\left(\frac{\partial F}{\partial x}\right)^2\right] dx \right\} dt = 0 \ . \qquad \text{II-6}$$

Y and F are related. The constraint equation relating them was developed in Chapter 2 as

$$\frac{S}{kn} \frac{\partial^2 F}{\partial x^2} - \frac{F}{AE} - h \frac{\partial^2 Y}{\partial x^2} = 0 \ . \qquad \text{II-7}$$

In a variational problem, two variables subject to a constraint may be treated as independent variables by introducing a new variable, say λ, multiplied by the constraint equation, into the function to be minimized. Thus the minimizing expression is

$$\delta \int_{t_1}^{t_2} \left\{ \int_o^L \left[\frac{3\rho A}{2}\left(\frac{\partial Y}{\partial t}\right)^2 - \frac{3EI}{2}\left(\frac{\partial^2 Y}{\partial x^2}\right)^2 - \frac{1}{AE} F^2 - \frac{S}{kn}\left(\frac{\partial F}{\partial x}\right)^2 - \lambda \left(\frac{S}{kn} \frac{\partial^2 F}{\partial x^2} + \frac{F}{AE} - h \frac{\partial^2 Y}{\partial x^2}\right)\right] dx \right\} dt = 0 \qquad \text{II-8}$$

or

$$\int_{t_1}^{t_2} \left\{ \int_{0}^{L} [3\rho A \frac{\partial Y}{\partial t}\delta(\frac{\partial Y}{\partial t}) - 3EI \frac{\partial^2 Y}{\partial x^2}\delta(\frac{\partial^2 Y}{\partial x^2}) - \frac{2}{AE} F\delta F - \frac{2S}{kn} \frac{\partial F}{\partial x}\delta(\frac{\partial F}{\partial x}) \right.$$

$$- \delta\lambda(\frac{S}{kn} \frac{\partial^2 F}{\partial x^2} - \frac{F}{AE} - h \frac{\partial^2 Y}{\partial x^2}) - \frac{\lambda S}{kn} \delta(\frac{\partial^2 F}{\partial x^2}) + \frac{\lambda}{AE} \delta F$$

$$\left. + h\lambda\ \delta(\frac{\partial^2 Y}{\partial x^2})]dx \right\} dt = 0 . \qquad \text{II-9}$$

For interchangeability of intergration and since δ and $\frac{\partial}{\partial t}$ and δ and $\frac{\partial}{\partial x}$ are commutative, then intergration by parts may be used. Thus

$$\int_{t_1}^{t_2} 3\rho A \frac{\partial Y}{\partial t} \delta(\frac{\partial Y}{\partial t})dt = 3\rho A \frac{\partial Y}{\partial t} \delta Y \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} 3\rho A \frac{\partial^2 Y}{\partial t^2} \delta Y dt$$

$$= - \int_{t_1}^{t_2} 3\rho A \frac{\partial^2 Y}{\partial t^2} \delta Y dt ,$$

since the initial and final configurations of the system are prescribed (i.e. $\delta Y = 0$ at t_1 and t_2).

Similarly

$$\frac{\partial^2 Y}{\partial x^2} \delta(\frac{\partial^2 Y}{\partial x^2}) = \frac{\partial}{\partial x}[\frac{\partial^2 Y}{\partial x^2} \delta(\frac{\partial Y}{\partial x})] - \frac{\partial}{\partial x}[\frac{\partial^3 Y}{\partial x^3} \delta Y] + \frac{\partial^4 Y}{\partial x^4} \delta Y ,$$

$$\frac{\partial F}{\partial x} \delta(\frac{\partial F}{\partial x}) = \frac{\partial}{\partial x}[\frac{\partial F}{\partial x} \delta F] - \frac{\partial^2 F}{\partial x^2} \delta F ,$$

$$\lambda\delta(\frac{\partial^2 F}{\partial x^2}) = \frac{\partial}{\partial x}[\lambda\delta(\frac{\partial F}{\partial x})] - \frac{\partial}{\partial x}[\frac{\partial\lambda}{\partial x}\delta F] + \frac{\partial^2\lambda}{\partial x^2}\delta F$$

and

$$\lambda\delta(\frac{\partial^2 Y}{\partial x^2}) = \frac{\partial}{\partial x}[\lambda\delta(\frac{\partial Y}{\partial x})] - \frac{\partial}{\partial x}[\frac{\partial\lambda}{\partial x}\delta Y] + \frac{\partial^2\lambda}{\partial x^2}\delta Y \ .$$

Applying the previous results to equation II-9 and collecting terms gives

$$\begin{aligned} - \int_{t_1}^{t_2} \Bigg\{ & \int_o^L [(3EI\frac{\partial^4 Y}{\partial x^4} - h\frac{\partial^2\lambda}{\partial x^2} + 3\rho A\frac{\partial^2 Y}{\partial t^2})\delta Y + (\frac{S}{kn}\frac{\partial^2\lambda}{\partial x^2} - \frac{2S}{kn}\frac{\partial^2 F}{\partial x^2} \\ & - \frac{\lambda}{AE} + \frac{2F}{AE})\delta F + (\frac{S}{kn}\frac{\partial^2 F}{\partial x^2} - \frac{F}{AE} - h\frac{\partial^2 Y}{\partial x^2})\delta\lambda]\, dx \\ & + (3EI\frac{\partial^2 Y}{\partial x^2} - h\lambda)\,\delta(\frac{\partial Y}{\partial x})\Big|_o^L - (3EI\frac{\partial^3 Y}{\partial x^3} - h\frac{\partial\lambda}{\partial x})\,\delta Y\Big|_o^L \\ & + \frac{2S}{kn}\lambda\delta(\frac{\partial F}{\partial x})\Big|_o^L + (\frac{2S}{kn}\frac{\partial F}{\partial x} - \frac{S}{kn}\frac{\partial\lambda}{\partial x})\,\delta F\Big|_o^L \Bigg\}\, dt = 0 \ . \end{aligned} \qquad \text{II-10}$$

The terms evaluated between the limits o and L provide boundary terms for the problem. The terms Y, F and λ may be varied arbitrarily and independently throughout the domain $o \leq x \leq L$. Then equation II-10 can be satisfied only if

$$3EI\frac{\partial^4 Y}{\partial x^4} - h\frac{\partial^2\lambda}{\partial x^2} + 3\rho A\frac{\partial^2 Y}{\partial t^2} = 0 \ , \qquad \text{II-11}$$

$$\frac{S}{kn}\frac{\partial^2\lambda}{\partial x^2} - \frac{2S}{kn}\frac{\partial^2 F}{\partial x^2} - \frac{\lambda}{AE} + \frac{2F}{AE} = 0 \qquad \text{II-12}$$

and

$$\frac{S}{kn}\frac{\partial^2 F}{\partial x^2} - \frac{F}{AE} - h\frac{\partial^2 Y}{\partial x^2} = 0 \ . \qquad \text{II-13}$$

From equation II-12 it is seen that

$$\lambda = 2F$$

and the governing equations for the three equal layer system are

$$3EI\frac{\partial^4 Y}{\partial x^4} - 2h\frac{\partial^2 F}{\partial x^2} + 3\rho A\frac{\partial^2 Y}{\partial t^2} = 0 \qquad \text{II-14}$$

and

$$\frac{S}{kn}\frac{\partial^2 F}{\partial x^2} - \frac{F}{AE} - h\frac{\partial^2 Y}{\partial x^2} = 0 \qquad \text{II-15}$$

with boundary conditions

$$\left(3EI\frac{\partial^2 Y}{\partial x^2} - 2hF\right)\delta\left(\frac{\partial Y}{\partial x}\right)\Big|_0^L = 0 \ ,$$

$$\left(3EI\frac{\partial^3 Y}{\partial x^3} - 2h\frac{\partial F}{\partial x}\right)\delta Y\Big|_0^L = 0$$

and

$$F\delta\left(\frac{\partial F}{\partial x}\right)\Big|_0^L = 0 \ . \qquad \text{II-16}$$

It is evident that the above equations and boundary conditions are equivalent to those developed in Chapter 2.

APPENDIX III

COMPUTER PROGRAMS

```
      PRCGRAM GRUNT
     1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     FIND EIGFNVALUES AS PER FORM
      DIMENSION CMEGASQ(30),CSTS(3),RCOTS(3),BCMAT(6,6)
     1,RNEG(30),RPOS1(30),RPOS2(30),DET(30)
      READ(5,100)NSTEP,DELTO
  100 FORMAT(I5,F10.5)
  999 READ(5,102)CCN1,CON2
 102  FORMAT(2F20.10)
      IF(EOF,5)104,103
  103 IFLAG=C
      READ(5,101)OMEGASC(1)
  101 FORMAT(F10.5)
      DO 15C I=2,NSTEP
      J=I-1
      OMEGASQ(I)=CMEGASQ(J)+DELTO
  150 CONTINUE
      CSTS(1)=CON1
      CSTS(2)=CON2
      DO 200 II=1,NSTEP
      CSTS(3)=OMEGASQ(II)
      CALL CUBIC(CSTS,RCCTS,IFLAG)
      RNEG(II)=RCCTS(1)
      RPOS1(II)=ROOTS(2)
      RPOS2(II)=RCOTS(3)
      IF(IFLAG.GT.0) GO TO 1CCO
      CALL FORM(CSTS,RCCTS,BCMAT)
      CALL DETERM(BCMAT,D)
      DET(II)=D
 2CC  CONTINUE
      WRITE(6,499)CON1,CON2
  499 FORMAT(1H1,2F20.5)
      WRITE(6,500)(OMEGASQ(I),DET(I),RNEG(I),RPOS1(I)
     1,RPOS2(I),I=1,NSTEP)
 5C0  FORMAT(1H0,5X,F15.7,5X,E15.5,5X,F15.7,5X,F15.7,
     15X,F15.7)
      GO TO 999
  104 WRITE(6,105)
  105 FORMAT(1H0,*THATS ALL FCLKS*)
      STOP
 1COO WRITE(6,1001) IFLAG,II
1C01  FORMAT(*ERRCR EXIT--IFLAG=*I2,*II=*I2)
      END
```

```
      SUBROUTINE CUBIC(CSTS,ROOTS,IFLAG)
C     COMPUTES ROOTS OF CUBIC
      DIMENSION CSTS(3),ROOTS(3)
      P=-9.*CSTS(1)
      Q=-CSTS(2)*CSTS(3)
      R=CSTS(1)*CSTS(2)*CSTS(3)
      AX=(3.*Q-P*P)/3.
      BX=(2.*P*P*P-9.*P*Q+27.*R)/27.
      DIS=BX*BX/4.+AX*AX*AX/27.
      IF(DIS)1,2,10
    1 PHI=(ACOS(-BX/2./SQRT(-AX*AX*AX/27.)))/3.
      XCOEFF=2.*SQRT(-AX/3.)
      CON=0.0174532925199433
      X1=XCOEFF*COS(PHI)-P/3.
      X2=XCOEFF*COS(PHI+120.*CON)-P/3.
      X3=XCOEFF*COS(PHI+240.*CON)-P/3.
      IF(X1)3,4,5
    3 ROOTS(1)=X1
      ROOTS(2)=X2
      ROOTS(3)=X3
      GO TO 11
    5 IF(X2)6,4,7
    6 ROOTS(1)=X2
      ROOTS(2)=X1
      ROOTS(3)=X3
      GO TO 11
    7 IF(X3)8,4,9
    8 ROOTS(1)=X3
      ROOTS(2)=X1
      ROOTS(3)=X2
      GO TO 11
    9 IFLAG=1
      GO TO 11
   10 IFLAG=2
      GO TO 11
    2 IFLAG=3
      GO TO 11
    4 IFLAG=4
   11 RETURN
      END
```

```
SUBROUTINE FORM(CSTS,ROOTS,BCMAT)
B.C. MATRIX FOR FIXED-FIXED BEAM
DIMENSION ROOTS(3),BCMAT(6,6),CSTS(3)
SINFUN=ABS(ROOTS(1))
GAM=SQRT(SINFUN)
ALPH=SQRT(ROOTS(2))
BET=SQRT(ROOTS(3))
C1=SINFUN*CSTS(1)/(SINFUN+CSTS(1))
C2=ROOTS(2)*CSTS(1)/(ROOTS(2)-CSTS(1))
C3=ROOTS(3)*CSTS(1)/(ROOTS(3)-CSTS(1))
E1=(GAM*SINFUN)*(SINFUN+9.*CSTS(1))/(SINFUN+CSTS(1))
E2=(ALPH*ROOTS(2))*(ROOTS(2)-9.*CSTS(1))/(ROOTS(2)-CSTS(1))
E3=(BET*ROOTS(3))*(ROOTS(3)-9.*CSTS(1))/(ROOTS(3)-CSTS(1))
CHALPH=.5*(EXP(ALPH)+EXP(-ALPH))
SHALPH=.5*(EXP(ALPH)-EXP(-ALPH))
CHBET=.5*(EXP(BET)+EXP(-BET))
SHBET=.5*(EXP(BET)-EXP(-BET))
BCMAT(1,1)=0.        $BCMAT(1,2)=1.
BCMAT(1,3)=0.        $BCMAT(1,4)=1.
BCMAT(1,5)=0.        $BCMAT(1,6)=1.
BCMAT(2,1)=GAM       $BCMAT(2,2)=0.
BCMAT(2,3)=ALPH      $BCMAT(2,4)=0.
BCMAT(2,5)=BET       $BCMAT(2,6)=0.
BCMAT(3,1)=0.
BCMAT(3,2)=SINFUN**2
BCMAT(3,3)=0.
BCMAT(3,4)=ROOTS(2)**2
BCMAT(3,5)=0.
BCMAT(3,6)=ROOTS(3)**2
BCMAT(4,1)=SIN(GAM)
BCMAT(4,2)=COS(GAM)
BCMAT(4,3)=SHALPH
BCMAT(4,4)=CHALPH
BCMAT(4,5)=SHBET
BCMAT(4,6)=CHBET
BCMAT(5,1)=GAM*COS(GAM)
BCMAT(5,2)=-GAM*SIN(GAM)
BCMAT(5,3)=ALPH*CHALPH
BCMAT(5,4)=ALPH*SHALPH
BCMAT(5,5)=BET*CHBET
BCMAT(5,6)=BET*SHBET
BCMAT(6,1)=SIN(GAM)*SINFUN**2
BCMAT(6,2)=COS(GAM)*SINFUN**2
BCMAT(6,3)=SHALPH*ROOTS(2)**2
BCMAT(6,4)=CHALPH*ROOTS(2)**2
BCMAT(6,5)=SHBET*ROOTS(3)**2
BCMAT(6,6)=CHBET*ROOTS(3)**2
RETURN
END
```

```
      SUBROUTINE DETERM(BCMAT,D)
C     COMPUTES DETERMINANT
      DIMENSION BCMAT(6,6),A(6,6)
      DO 200 I=1,6
      DO 200 J=1,6
  200 A(I,J)=BCMAT(I,J)
      D=1.
      K=1
    1 CONTINUE
      KK=K+1
      IS=K
      IT=K
      B=ABS(A(K,K))
      DO 2 I=K,6
      DO 2 J=K,6
      IF(ABS(A(I,J))-B)2,2,21
   21 IS=I
      IT=J
      B=ABS(A(I,J))
    2 CONTINUE
      IF(IS-K)3,3,31
   31 DO 32 J=K,6
      C=A(IS,J)
      A(IS,J)=A(K,J)
   32 A(K,J)=-C
    3 CONTINUE
      IF(IT-K)4,4,41
   41 DO 42 I=K,6
      C=A(I,IT)
      A(I,IT)=A(I,K)
   42 A(I,K)=-C
    4 CONTINUE
      D=A(K,K)*D
      IF(A(K,K))5,71,5
    5 CONTINUE
      DO 6 J=KK,6
      A(K,J)=A(K,J)/A(K,K)
      DO 6 I=KK,6
      W=A(I,K)*A(K,J)
      A(I,J)=A(I,J)-W
      IF(ABS(A(I,J))-.0001*ABS(W))61,6,6
   61 A(I,J)=0.
    6 CONTINUE
      K=KK
      IF(K-6)1,70,1
   70 D=A(6,6)*D
   71 RETURN
      END
```

```
      PROGRAM MODES
     1(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
C     DETERMINES MODE SHAPE AS PER FORM
      DIMENSION A(6),B(5,6),Y(30),D(30),XV(5),X(5),
     1C(5,6),YN(30)
      READ(5,100)NSTEP,DELX
  100 FORMAT(I5,F10.5)
      KER=0
      D(1)=0.
      DO 10 I=2,NSTEP
      J=I-1
      D(I)=D(J)+DELX
   10 CONTINUE
  997 READ(5,101)GAM2,ALF2,BET2
  101 FORMAT(3F20.10)
      IF(EOF,5)999,998
  998 READ(5,102)C1,C2
  102 FORMAT(2F10.3)
      WRITE(6,200)C1,C2
  200 FORMAT(1H1,*   C1=*F10.3,*   C2=*F10.3)
      CALL FORM(B,GAM2,ALF2,BET2,C1,C2)
      CALL LINSOLV(B,C,X)
      A(1)=-1.
      DO 20 I=2,6
      J=I-1
      A(I)=X(J)
   20 CONTINUE
      CALL DEFLEX(Y,D,A,GAM2,ALF2,BET2,NSTEP)
      DO 30 I=1,NSTEP
      YN(I)=Y(I)/Y(1)
   30 CONTINUE
      WRITE(6,201)(A(I),I=1,6)
  201 FORMAT(1H0,*A'S ARE*6E20.5)
      WRITE(6,202)(D(I),Y(I),YN(I),I=1,NSTEP)
  202 FORMAT(3E30.7)
      GO TO 997
  999 WRITE(6,203)
  203 FORMAT(1H0,*AND SO IT IS*)
      END
```

```
      SUBROUTINE FORM(B,GAM2,ALF2,BET2,C1,C2)
C     MODE SHAPE MATRIX FOR FREE-FREE BEAM
      DIMENSION B(5,6)
      GAM=SQRT(GAM2)
      ALF=SQRT(ALF2)
      BET=SQRT(BET2)
      CC1=GAM2*C1/(GAM2+C1)
      CC2=ALF2*C1/(ALF2-C1)
      CC3=BET2*C1/(BET2-C1)
      E1=GAM*GAM2*(GAM2+9.*C1)/(GAM2+C1)
      E2=ALF*ALF2*(ALF2-9.*C1)/(ALF2-C1)
      E3=BET*BET2*(BET2-9.*C1)/(BET2-C1)
      SHALF=.5*(EXP(ALF)-EXP(-ALF))
      CHALF=.5*(EXP(ALF)+EXP(-ALF))
      SHBET=.5*(EXP(BET)-EXP(-BET))
      CHBET=.5*(EXP(BET)+EXP(-BET))
      B(1,1)=-GAM2
      B(1,2)=C.
      B(1,3)=ALF2
      B(1,4)=C.
      B(1,5)=BET2
      B(1,6)=C.
      B(2,1)=C.
      B(2,2)=E2
      B(2,3)=C.
      B(2,4)=E3
      B(2,5)=C.
      B(2,6)=-E1
      B(3,1)=CC1
      B(3,2)=C.
      B(3,3)=CC2
      B(3,4)=C.
      B(3,5)=CC3
      B(3,6)=C.
      B(4,1)=-GAM2*COS(GAM)
      B(4,2)=ALF2*SHALF
      B(4,3)=ALF2*CHALF
      B(4,4)=BET2*SHBET
      B(4,5)=BET2*CHBET
      B(4,6)=-GAM2*SIN(GAM)
      B(5,1)=E1*SIN(GAM)
      B(5,2)=E2*CHALF
      B(5,3)=E2*SHALF
      B(5,4)=E3*CHBET
      B(5,5)=E3*SHBET
      B(5,6)=-E1*COS(GAM)
      RETURN
      END
```

```
      SUBROUTINE LINSOLV(B,C,X)
C     CONTROL FOR LSSDP
      DIMENSION B(5,6),C(5,6),X(5),XV(5)
      DO 10 I=1,5
      DO 10 J=1,6
   10 C(I,J)=B(I,J)
      CALL LSSDP(C,5,6,XV,KER)
      IF (KER .EQ. 2) STOP
      DO 20 I=1,5
   20 X(I) = XV(I)
      RETURN
      END
```

```
      SUBROUTINE LSSDP(C,N,NPM,X,KER)
C     SOLUTION OF LINEAR SYSTEM BY GAUSS ELIMINATION
      DIMENSION C(N,NPM),X(N)
      EQUIVALENCE (R,S),(RATIO,V)
      R = 0.0
      DO 111 J=1,N
      Z= 0.0
      DO 110 K=1,N
      V = C(J,K)
  110 Z = Z + ABS(V)
      IF(R.GE.Z) GO TO 111
      R = Z
  111 CONTINUE
      EX = (1.0E-20)*R
      DO 34 L=1,N
      Z = 0.0           $    KP=0
      DO 12 K=L,N
      AER = C(K,L)             $    AER=ABS(AER)
      IF(Z.GE.AER) GO TO 12
      Z = AER           $    KP=K
   12 CONTINUE
      IF (L.GE.KP) GO TO 20
      DO 14 J = L,NPM
      S = C(L,J)
      C(L,J) = C(KP,J)
   14 C(KP,J)=S
   20 AER = C(L,L)             $    AER=ABS(AER)
      IF(AER.LE.EX) GO TO 50
      IF(L.GE.N) GO TO 40
      LP1 = L+1
      DO 34 K=LP1,N
      AER = C(K,L)             $    AER=ABS(AER)
      IF(AER) 32,34,32
   32 RATIO=C(K,L)/C(L,L)
      DO 33 J=LP1,NPM
   33 C(K,J)=C(K,J)-RATIO*C(L,J)
   34 CONTINUE
   40 DO 44 I=1,N
      S = 0.0D0
      II = NPM - I
      IF(II.GE.N) GO TO 43
      IIP1 = II + 1
      DO 42 K=IIP1,N
   42 S=S+C(II,K)*X(K)
   43 RATIO=C(II,NPM)
   44 X(II)=(RATIO-S)/C(II,II)
      KER = 1
      RETURN
   50 KER = 2
      PRINT 101
  101 FORMAT(/36H MATRIX SINGULAR IN SUBROUTINE LSSDP /)
      RETURN
      END
```

```
      SUBROUTINE DEFLEX(Y,D,A,GAM2,ALF2,BET2,NSTEP)
      GIVES SHAPE
      DIMENSION D(30),Y(30),A(6)
      GAM=SQRT(GAM2)
      ALF=SQRT(ALF2)
      BET=SQRT(BET2)
      DO 5 I=1,NSTEP
      GX=GAM*D(I)
      AX=ALF*D(I)
      BX=BET*D(I)
      R1=A(1)*SIN(GX)+A(2)*COS(GX)
      R2=.5*(EXP(AX)*(A(3)+A(4))+EXP(-AX)*(A(4)-A(3)))
      R3=.5*(EXP(BX)*(A(5)+A(6))+EXP(-BX)*(A(6)-A(5)))
      Y(I)=R1+R2+R3
    5 CONTINUE
      RETURN
      END
```

```
      PROGRAM SIMDIF
      FINITE DIFFERENCE CHECK ON SIMPLE SYSTEM
      MATRIX = CDC CANNED PROGRAM
      DIMENSION OF MATRICES MUST EQUAL N
      DIMENSION G(25,25),H(25,25),U(25,25),D(25,25)
  999 READ(5,1)C1,C2,BL,SECT,N
    1 FORMAT(4F10.5,I5)
      DO 5 I=1,N
      DO 5 J=1,N
      G(I,J)=0.
      H(I,J)=0.
      D(I,J)=0.
      U(I,J)=0.
      U(I,I)=1.
    5 CONTINUE
      INVERT=10
      MULT=20
      IGEN=3
      SQ=(BL/SECT)**2
      FOUR=SQ**2
      SIX=SQ*FOUR
      R1=9.*C1*SQ+6.
      R2=36.*C1*SQ+15.
      R3=C2*FOUR
      R4=54.*C1*SQ+20.
      R5=2*R3+C1*C2*SIX
      K=N-1
      DO 10 I=2,K
      H(I,I)=-R5
      J=I-1
      JJ=I+1
      H(I,J)=R3
      H(I,JJ)=H(I,J)
   10 CONTINUE
      H(1,1)=-R5
      H(N,N)=H(1,1)
      H(N,K)=H(2,1)
      H(1,2)=H(2,1)
      DO 20 I=4,N
      G(I,I)=-R4
      J=I-1
      JJ=I-2
      JJJ=I-3
      G(I,J)=R2
      G(I,JJ)=-R1
      G(I,JJJ)=1.
   20 CONTINUE
```

```
      G(1,1)=R1-R4
      G(N,N)=G(1,1)
      G(2,1)=R2-1.
      G(N,K)=G(2,1)
      G(3,1)=-R1
      G(3,2)=R2
      G(2,2)=-R4
      G(3,3)=G(2,2)
      DO 30 I=1,N
      DO 30 J=1,N
      G(I,J)=G(J,I)
 30   CONTINUE
      WRITE(6,100)
      WRITE(6,99)((G(I,J),J=1,N),I=1,N)
      WRITE(6,100)
      WRITE(6,99)((U(I,J),J=1,N),I=1,N)
      WRITE(6,100)
      WRITE(6,99)((H(I,J),J=1,N),I=1,N)
  99  FORMAT(7E15.5)
      CALL MATRIX(INVERT,N,N,0,G,N,DET)
      CALL MATRIX(MULT,N,N,N,G,N,H,N,D,N)
      CALL MATRIX(IGEN,N,0,1,D,N,U,N,0,4)
      WRITE(6,100)
100   FORMAT(1H1)
      WRITE(6,101)C1,C2,BL,SECT
101   FORMAT(1H0,*C1=*F10.3,*C2=*F10.3,*LENGTH=*F10.3,
     1*SECTIONS=*F10.3,///)
      WRITE(6,102)(D(I,1),I=1,4)
102   FORMAT(1H0,E15.7)
      GO TO 999
      END
```