DISSERTATION

RAMSEY REGIONS AND SIMPLICIAL HOMOLOGY TABLES FOR GRAPHS

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Christopher Austin Frederick

Department of Mathematics

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY CHRISTOPHER AUSTIN FREDERICK ENTITLED "RAMSEY REGIONS AND SIMPLICIAL HOMOLOGY TABLES FOR GRAPHS" BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Committee on Graduate Work

Dr. Ross McConnell

Dr. Robert Liebler

Dr. Anton Betten

Adviser: Dr. Chris Peterson

Department Head: Dr. Gerhard Dangelmayr

ABSTRACT OF DISSERTATION

RAMSEY REGIONS AND SIMPLICIAL HOMOLOGY TABLES FOR GRAPHS

Ramsey Theory is the investigation of edge-colored graphs which force a monochromatic subgraph. We devise a way of breaking certain Ramsey Theory problems into "smaller"pieces so that information about Ramsey Theory can be gained without solving the entire problem, (which is often difficult to solve).

Next the work with Ramsey Regions for graphs is translated into the language of hypergraphs. Theorems and techniques are reworked to fit appropriately into the setting of hypergraphs.

The work of persistence complex on large data sets is examined in the setting of graphs. Various simplicial complexes can be assigned to a graph. For a given simplicial complex the persistence complex can be constructed, giving a highly detailed graph invariant. Connections between the graph and persitance complex are investigated.

Christopher Austin Frederick
Department of Mathematics
Colorado State University
Fort Collins, Colorado 80523
Fall 2008

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Chapter 1

RAMSEY REGIONS

1.1 Introduction

The work that follows (Chapter 1) appeared in Discrete Mathematics Vol 308 Issue 18 [?].

The motivation behind the material in Chapter one is to place the large and difficult problems (see [?]) that occur in classical Ramsey Theory in a setting that allows smaller problems to interconnect and build and allows algebra to contribute. A specific class of problems in classical Ramsey theory are stated in the context of colorings of complete graphs. By embedding these problems into a setting which includes all complete multipartite graphs, we see a more discretized set of problems and a stronger interconnectedness which allows solutions of one problem to more directly aid in the solution of nearby problems. Furthermore, the interconnectedness allows the creation of a Ramsey Region. In Chapter one, we develop the basic theory and tools for studying Ramsey Regions and we associate to each region an algebra.

Classical Ramsey Theory is actively studied using various algebraic and combinatorial techniques. Many of the values and bounds attained in classical Ramsey Theory can be found in [?].

I begin by defining a Ramsey Region. Next the tools that relate information about one Ramsey Region to information about similar Ramsey Regions (Proposition 1.2.1, related corollaries, and Proposition 1.2.2) are developed. Then the underlying structure (shape) of the Ramsey Region is determined (Theorem 1.2.1). Work and results useful in forming the Ramsey Region can be found in [?, ?, ?, ?].

Let $(T_1, T_2, \ldots, T_c)$ be a fixed c-tuple of sets of graphs (i.e. each T_i is a set of graphs). Let $R(c, n, (T_1, T_2, \ldots, T_c))$ denote the set of all n-

tuples, $(a_1, a_2, \ldots, a_n)$, such that every c-coloring of the edges of the complete multipartite graph, $K_{a_1,a_2,\ldots,a_n}$, forces a monochromatic subgraph of color i from the set T_i (for at least one i). If $\mathbb{N}$ denotes the set of non-negative integers, then $R(c, n, (T_1, T_2, \ldots, T_c)) \subseteq \mathbb{N}^n$. We call such a subset of $\mathbb{N}^n$ a "Ramsey region". An application of Ramsey's Theorem shows that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty for $n \gg 0$. For a given c-tuple, $(T_1, T_2, \ldots, T_c)$, known results in Ramsey theory help identify values of n for which the associated Ramsey regions are non-empty and help establish specific points that are in such Ramsey regions. In this paper, we develop the basic theory and some of the underlying algebraic structure governing these regions.

Ramsey theory dates back 75 years to the following theorem:

Theorem 1.1.1. *(Ramsey) [?] Let r, k, l be given positive integers. There exists a positive integer n with the following property. If the k-subsets of an n element set are colored with r colors then there exists an l element set all of whose k-subsets are the same color.*

For a given r, k, l it is an interesting (and hard) problem to find the smallest value of n guaranteed to exist by Ramsey's theorem. The theorem has many corollaries guaranteeing the existence of substructures under various conditions. If k is set equal to 2 then Ramsey's theorem is a theorem in graph theory. It states that for n sufficiently large, any r coloring of the edges of K_n contains a monochromatic subgraph isomorphic to K_l. Since any graph, G, embeds in some complete graph, the theorem also implies that for n sufficiently large, any r coloring of the edges of K_n contains a monochromatic subgraph isomorphic to G.

In this paper, $K_{a_1,a_2,\ldots,a_t}$ will denote the complete t-partite graph on sets of vertices of size $a_1, a_2, \ldots, a_t$ and K_n will denote the complete graph on n vertices (thus $K_n = K_{(1,1,\ldots,1)}$). Let G be an arbitrary graph. If the vertices of G can be colored with t colors such that adjacent vertices have different colors then G can be embedded into a complete t-partite graph. The smallest t such that G can be embedded into a complete t-partite graph is called the chromatic number of G and is denoted by $\chi(G)$.

Let $(T_1, T_2, \ldots T_c)$ be a c-tuple of sets of graphs (i.e. each T_i is a *set* of graphs). Let $R(c, n, (T_1, T_2, \ldots, T_c))$ denote the set of all n-tuples such that every c-coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph of color i from the set T_i (for some i). Ramsey's Theorem guarantees that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty for $n \gg 0$. More precisely, Ramsey's Theorem guarantees that the n-tuple consisting entirely of 1's is an element of $R(c, n, (T_1, T_2, \ldots, T_c))$ provided that $n \gg 0$. The set of all n-tuples in $R(c, n, (T_1, T_2, \ldots, T_c))$ is called a **Ramsey Region**. The goal of this paper is to develop the basic theory of Ramsey regions. Let M denote the minimum value of the chromatic numbers of the graphs in the various T_i's. M provides a lower bound on n such that $R(c, n, (T_1, T_2, \ldots, T_c))$ is non-empty. Known results in Ramsey theory can be used to give upper bounds on n such that the n-tuple consisting entirely of 1's lies in $R(c, n, (T_1, T_2, \ldots, T_c))$. In general, these bounds are far from being sharp. It is important to note that any given Ramsey region can be described completely by a *finite* list of n-tuples (even though a non-empty Ramsey region will have an infinite number of points).

1.2 Main definitions and theorems

For this entire section, $K_{a_1,a_2,\ldots,a_n}$ will denote a complete multipartite graph and $A = (T_1, T_2, \ldots, T_c)$ will be an ordered c-tuple of sets of graphs. All graphs are assumed to have no multiple edges and no loops. $R(c, n, A)$ will denote the set of all n-tuples such that every c-coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph of color i which is isomorphic to a graph from the set T_i (for at least one value of i). If $T = T_1 = T_2 = \cdots = T_c$ then we write $R(c, n, T)$ instead of $R(c, n, (T_1, T_2, \ldots, T_c))$.

Proposition 1.2.1. *Suppose* $K_{a_1,a_2,\ldots,a_n} \subseteq K_{b_1,b_2,\ldots,b_m}$ *then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \implies (b_1, b_2, \ldots, b_m) \in R(c, m, A).$$

Proof. Fix an injection $K_{a_1,a_2,\ldots,a_n} \hookrightarrow K_{b_1,b_2,\ldots,b_m}$. In coloring the edges of $K_{b_1,b_2,\ldots,b_m}$ with c colors, you induce a coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ with c colors. Thus, if there exists a monochromatic subgraph of $K_{b_1,b_2,\ldots,b_m}$ of color i which is isomorphic to a given graph G then the induced coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ will also contain a monochromatic subgraph of color i which is isomorphic to G. □

Corollary 1.2.1. *Let* S_n *denote the symmetric group on* n *elements and suppose* $\pi \in S_n$, *then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \iff (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}) \in R(c, n, A).$$

Proof. Follows from Proposition 1.2.1 using $K_{a_1,a_2,\ldots,a_n} \simeq K_{a_{\pi(1)},a_{\pi(2)},\ldots,a_{\pi(n)}}$. □

Corollary 1.2.2. *If* $(a_1, a_2, \ldots, a_n)$ *and* $(b_1, b_2, \ldots, b_n)$ *are* n*-tuples such that* $b_i \geq a_i$ *for* $1 \leq i \leq n$ *then*

$$(a_1, a_2, \ldots, a_n) \in R(c, n, A) \implies (b_1, b_2, \ldots, b_n) \in R(c, n, A).$$

Proof. Follows from Proposition 1.2.1 using $K_{a_1,a_2,\ldots,a_n} \subseteq K_{b_1,b_2,\ldots,b_n}$. □

Corollary 1.2.3. *If $(a_1, a_2, \ldots, a_n)$ is an n-tuple and b satisfies $0 \leq b \leq a_1$ then*

$$(a_1, \ldots, a_n) \in R(c, n, A) \implies (a_1 - b, a_2, \ldots, a_n, b) \in R(c, n+1, A).$$

Proof. Follows from Proposition 1.2.1 using $K_{a_1,a_2,\ldots,a_n} \subseteq K_{a_1-b,a_2,\ldots,a_n,b}$. □

Definition 1.2.1. *Suppose $(a_1, a_2, \ldots, a_n) \in R(c, n, A)$. If there exists an i such that lowering a_i by 1 gives an n-tuple which is not in $R(c, n, A)$ then we call $(a_1, a_2, \ldots, a_n)$ a* **boundary point**. *A boundary point is called a* **vertex** *if lowering any a_i by 1 gives an n-tuple which is not in $R(c, n, A)$. The vertex is called a* **fundamental vertex** *if $a_1 \geq a_2 \geq \cdots \geq a_n$.*

Proposition 1.2.2. *(i) Let $A = (T_1, T_2, \ldots, T_c)$ and $B = (V_1, V_2, \ldots, V_c)$ be c-tuples of sets of graphs. If for each i, each element of V_i contains as a subgraph some element of T_i then $R(c, n, B) \subseteq R(c, n, A)$.*

(ii) If $\pi \in S_c$ then $R(c, n, (T_1, T_2, \ldots, T_c)) = R(c, n, (T_{\pi(1)}, T_{\pi(2)}, \ldots, T_{\pi(c)}))$.

(iii) Let $A = (T_1, T_2, \ldots, T_{c-1}, T_c \cup T'_c), B = (T_1, T_2, \ldots, T_{c-1}, T_c)$ and $C = (T_1, T_2, \ldots, T_{c-1}, T'_c)$ then $R(c, n, A) = R(c, n, B) \cup R(c, n, C)$.

Proof. For Part (i), note that if every c-coloring of the edges of $K_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph as determined by B then it also forces a monochromatic subgraph as determined by A. Part (ii) and Part (iii) are clear. □

As a consequence of Proposition 1.2.2 iii), it is enough to understand $R(c, n, A)$ where A is a c-tuple of graphs (as opposed to A being a c-tuple of sets of graphs).

Theorem 1.2.1. *Let k be a field and let $S = k[x_1, x_2, \ldots, x_n]$. Define a map $\phi : R(c,n,A) \to S$ by $\phi((a_1, a_2, \ldots, a_n)) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$. Let $I(R(c,n,A))$ be the homogeneous ideal in S generated by the image of ϕ. Let V be the set of vertices of $R(c,n,A)$ and let $I(V)$ be the homogeneous ideal in S generated by $\phi(V)$. Then*

(i) $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in I(R(c,n,A)) \iff (a_1, a_2, \ldots, a_n) \in R(c,n,A)$

(ii) $I(R(c,n,A)) = I(V)$.

Proof. (i) If $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in I(R(c,n,A))$ then $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is an S-linear combination of elements in $\phi(R(c,n,A))$. As a consequence of Corollary 1.2.2, if $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in \phi(R(c,n,A))$ and if M is a monomial then $M \cdot x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in \phi(R(c,n,A))$. Thus the degree d part of $I(R(c,n,A))$ is equal to the span of the degree d part of $\phi(R(c,n,A))$.

(ii) By part (i), $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in I(R(c,n,A)) \iff (a_1, a_2, \ldots, a_n) \in R(c,n,A)$. Given $(a_1, a_2, \ldots, a_n) \in R(c,n,A)$, decrease a_n as much as possible while remaining inside $R(c,n,A)$ then decrease a_{n-1} as much as possible while remaining inside $R(c,n,A)$ then do the same for a_{n-2} on down to a_1. At this point, lowering any of the components any further will cause you to leave $R(c,n,A)$. In other words, you are at a vertex. In terms of S, this is equivalent to factoring $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} = M \cdot \phi(v)$ where M is a monomial and v is a vertex. But this shows $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \in I(V)$. □

Let S_n denote the symmetric group on n elements. For each $\pi \in S_n$ we get a map $\pi_S : S \to S$ by $\pi_S(x_i) = x_{\pi(i)}$. In this way, S_n acts on S. Furthermore, each element of $R(c,n,A)$ extends to an element of $R(c,n+1,A)$ by sending $(a_1, a_2, \ldots, a_n)$ to $(a_1, a_2, \ldots, a_n, 0)$. Thus there is a natural embedding $f : R(c,n,A) \hookrightarrow R(c,n+1,A)$. Since $R(c,n+1,A)$ is

S_{n+1}-invariant, the smallest S_{n+1} invariant set containing $f(R(c,n,A))$ also sits inside $R(c,n+1,A)$.

Definition 1.2.2. *(i)* $V_n(c,A)$ *denotes the vertices of* $R(c,n,A)$.

(ii) $F_n(c,A)$ *denotes the fundamental vertices of* $R(c,n,A)$.

(iii) If $T \subseteq k[x_1,x_2,\ldots,x_n]$ *then* $S_n(T)$ *will denote the smallest* S_n*-invariant set containing* T.

(iv) Z_i *denotes the* i*-tuple* $(1,1,\ldots,1)$.

(v) If $f=(a_1,a_2,\ldots,a_n)$ *then define* $[f]=\Sigma_{i=1}^{n}a_i$.

Proposition 1.2.3. *(i)* $S_n(F_n(c,A))=V_n(c,A)$.

(ii) There exists an i *such that* $Z_i \in F_i(c,A)$.

(iii) $F_n(c,A) \hookrightarrow F_{n+1}(c,A)$.

(iv) $\lim_{n\to\infty} F_n(c,A)$ *exists.*

Proof. Part (i) is clear. Part (ii) follows from Ramsey's Theorem. Part (iii) follows from the fact that $R(c,n,A)$ injects into $R(c,n+1,A)$ by the map $(a_1,a_2,\ldots,a_n) \to (a_1,a_2,\ldots,a_n,0)$ and this injection maps fundamental vertices to fundamental vertices. Part (iv) follows from Part (ii) and its implication that $Z_i \in F_i(c,A)$ implies $F_i(c,A) \simeq F_{i+t}(c,A)$ for all $t>0$. □

Definition 1.2.3. *Let* $v=(a_1,a_2,\ldots,a_n)$. *Let* $\bar{v}$ *denote the tuple obtained from* v *by removing any* a_i *that are equal to zero. Let* $\overline{F_n(c,A)}=\{\bar{v}|v \in F_n(c,A)\}$. *Let* $\mathfrak{F}(c,A)=\lim_{n\to\infty}\overline{F_n(c,A)}$. *We call* $\mathfrak{F}(c,A)$ *the* **Full Set of Fundamental Vertices** *for* c *and* A. *From* $\mathfrak{F}(c,A)$ *we can reconstruct* $R(c,n,A)$ *for any value of* n.

For a fixed c and A, define the function $HF(c,A)(n) = |F_n(c,A)|$ and the sequence $HF(c,A) = |F_0(c,A)|, |F_1(c,A)|, |F_2(c,A)|, \ldots$. As a consequence of Proposition 1.2.3, $HF(c,A)$ will be a bounded, monotone increasing function which stabilizes at $|\mathfrak{F}(c,a)|$. Define $P_{bottom}(c,A) = max\{n|HF(c,A)(n) = 0\}$ and $P_{top}(c,A) = min\{n|HF(c,A) = |\mathfrak{F}(c,a)|\}$.

Proposition 1.2.4. *Fix c and A. If $P_{bottom}(c,A) \leq n \leq P_{top}(c,A)$, then the function $HF(c,A)(n)$ is a strictly increasing function of n.*

Proof. Given $f = (a_1, a_2, \ldots, a_t)$, define $||f||$ to be the number of a_i that are not equal to zero. Assume that $Z_i \notin F_i(c,A)$ for any $i \leq n$. We need to show there exists a vertex $g \in F_{n+1}(c,A)$ such that $||g|| = n+1$ (for this would show that the number of elements in $F_{n+1}(c,A)$ is strictly larger than the number of elements in $F_n(c,A)$). Let $r = min\{[h]|h \in F_n(c,A)\}$. Pick an element $f \in F_n(c,A)$ such that $[f] = r$ (see Definition 1.2.2). Using Corollary 1.2.3, f can be used to produce a point, $f' \in R(c, n+1, A)$ with $||f'|| = min\{r, n+1\}$. If $r < n+1$ then $\bar{f}' = Z_r$ which contradicts our assumptions. If $r \geq n+1$ then $||f'|| = n+1$. If $f' \in V_{n+1}(c,A)$ then by Corollary 1.2.1, we can use f' to make a fundamental vertex $g \in F_{n+1}(c,A)$ with $||g|| = n+1$. If $f' \notin V_{n+1}(c,A)$, then by Theorem 1.2.1, $\phi(f') = M \cdot \phi(v)$ with v a vertex. Since $[v] < [f'] = [f]$ and since f was chosen from $F_n(c,A)$ such that $[f]$ was as small as possible, we can conclude that $v \notin F_n(c,A)$. Since $v \notin F_n(c,A)$ and $\phi(f') = M \cdot \phi(v)$ and since $||f'|| = n+1$, we can conclude that $v \in F_{n+1}(c,A)$ and that $||v|| = n+1$. Thus, while the map $F_n(c,A) \hookrightarrow F_{n+1}(c,A)$ is injective, it is not surjective hence $|F_{n+1}(c,A)| > |F_n(c,A)|$. □

Let $G_1, G_2, \ldots, G_c$ be graphs. Let $A = (G_1, G_2, \ldots, G_c)$. Let $\mathbf{R}(A)$ denote the smallest n such that every c-coloring of the edges of K_n necessarily

contains a monochromatic graph of color i which is isomorphic to G_i for at least one value of i. Thus $\mathbf{R}(A)$ denotes the standard c-color Ramsey number avoiding G_i in color i. We have the following proposition:

Proposition 1.2.5. $\mathbf{R}(A) \leq min\{[f] \ | f \in F_i(c,A)\}$ *for every* i *and* $\mathbf{R}(A) = min\{[f] \ | f \in \mathfrak{F}(c,A)\} = P_{top}(c,A) = min\{i | Z_i \in F_i(c,A)\}$.

Proof. The proof follows immediately from Corollary 1.2.3 and Proposition 1.2.3. □

Example 1.2.1. *Let* $A = (K_{2,2}, K_{2,1})$, *then*

$F_0(2,A) = F_1(2,A) = \emptyset$

$F_2(2,A) = \{(4,2)\}$

$F_3(2,A) = \{(4,2,0),(2,2,2),(3,2,1),(4,1,1)\}$

$F_4(2,A) = \{(4,2,0,0),(2,2,2,0),(3,2,1,0),(4,1,1,0),(1,1,1,1)\}$

$\mathfrak{F}(2,A) = \{(4,2),(2,2,2),(3,2,1),(4,1,1),(1,1,1,1)\}$

$P_{bottom}(2,A) = 1,\ P_{top}(2,A) = 4$

$HF(2,A) = 0,0,1,4,5,5,\ldots$

$\mathbf{R}(A) = 4.$

Definition 1.2.4. *Let* G *be a graph with vertex set* V *and edge set* E. *Let* $v \in V$. *Form a new graph,* G', *from* G *by adding a new vertex* v' *and by adding new edges by the rule:* v' *is connected to* $w \in V$ *if and only if* v *is connected to* w. *Then* G' *is said to be obtained from* G *by a* **neighborhood duplication**.

Theorem 1.2.2. *If* G *is a graph and if* G' *is obtained from* G *by a neighborhood duplication, then* $\chi(G) = \chi(G')$. *Furthermore,* $K_n \subseteq G \iff K_n \subseteq G'$.

Proof. It is clear that $K_n \subseteq G \implies K_n \subseteq G'$. We need to show that $K_n \not\subseteq G \implies K_n \not\subseteq G'$. Let v be the vertex that was duplicated and let v' be the new vertex. If $K_n \not\subseteq G$ and $K_n \subseteq G'$ then the copy of K_n in G' must contain both v and v'. But v and v' are not connected. In a complete graph every pair of vertices is connected. Therefore $K_n \not\subseteq G \implies K_n \not\subseteq G'$.

To prove that $\chi(G) = \chi(G')$, note that if a vertex coloring of G avoids adjacent vertices having the same color then by coloring v' the same color as v, we have colored the vertices of G' so that adjacent vertices don't have the same color. □

Corollary 1.2.4. $\mathfrak{F}(c, (K_{b_1}, K_{b_2}, \ldots, K_{b_c})) = \{Z_b\}$ *where* Z_b *is the* b*-tuple of 1's with* $b = \mathbf{R}(K_{b_1}, K_{b_2}, \ldots, K_{b_c})$.

Proof. Let $j < \mathbf{R}(K_{b_1}, K_{b_2}, \ldots, K_{b_c})$. Then there exists a coloring of K_j containing no K_{b_i} of color i (for any i). By a sequence of neighborhood duplications, we can transform K_j into $K_{a_1,a_2,\ldots,a_j}$ for any j-tuple $(a_1, a_2, \ldots, a_j)$ (with strictly positive coordinates). If we duplicate both the neighborhoods and the corresponding edge coloring, then we obtain an edge coloring of $K_{a_1,a_2,\ldots,a_j}$ containing no K_{b_i} of color i for any i (by Proposition 1.2.2). Thus $F_j(c, (K_{b_1}, K_{b_2}, \ldots, K_{b_c})) = \emptyset$. If $j = \mathbf{R}(K_{b_1}, K_{b_2}, \ldots, K_{b_c})$ then $Z_j \in R(c, j, (K_{b_1}, K_{b_2}, \ldots, K_{b_c}))$ thus there is only one fundamental vertex. □

1.3 Two dimensional Ramsey regions

This section is concerned with properties of Ramsey regions of the form $R(c, 2, A)$. We call these *two-dimensional Ramsey regions* since they consist of 2-tuples. Let the vertices of a complete bipartite graph $K_{a,b}$ be represented by $x_1, x_2, \ldots, x_a$ and $y_1, y_2, \ldots, y_b$. Any c-coloring of the edges

of $K_{a,b}$ can be represented by an $a \times b$ matrix whose ij^{th} entry is the color of the edge connecting x_i and y_j. If A is a c-tuple of complete bipartite graphs, then determining $R(c, 2, A)$ is equivalent to determining for what values of r, s is every $r \times s$ matrix, whose entries are one of c different colors, forced to contain an appropriate sized monochromatic rectangle. If only two colors are used, then we can use the symbols 0 and 1 to represent the two colors. If $A = (K_{a,b}, K_{c,d})$ is a 2-tuple of complete bipartite graphs then determining $R(2, 2, A)$ is equivalent to determining for what values of r, s is every $r \times s$ 0,1-matrix forced to contain either an ($a \times b$ or $b \times a$) submatrix of 1's or a ($c \times d$ or $d \times c$) submatrix of 0's. Standard constructions in Design theory can be used to produce large 0,1-matrices which do not contain monochromatic sub-rectangles.

Definition 1.3.1. *Let t, k, v, λ be integers with $t < k < v$ and $\lambda > 0$. Then a $t - (v, k, \lambda)$* **design** *is a collection of k-element subsets (called* **blocks***) of a v-element set (called* **points***) such that every t-element subset of the v-element set is contained in exactly λ of the k-element sets.*

The point-block incidence matrix of a design can be reinterpreted as the reduced adjacency matrix of a bipartite graph or as a coloring of the edges of a complete bipartite graph with 2 colors. Properties of the design imply various properties of the 0,1-matrix. This is illustrated in the following:

Example 1.3.1. *Consider the set consisting of 3-subsets of $\{1, 2, 3, 4, 5, 6, 7\}$, $Y = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$. Y is a $2 - (7, 3, 1)$ design (it is also a Steiner triple system of order 7). The*

incidence matrix of Y *is the matrix* $M_Y = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$. *Since* Y *is a* $2-(7,3,1)$ *design, there can be no* 2×2 *submatrix of 1's. It is easy to check that there are no* 2×3 *nor* 3×2 *submatrices of 0's. Thus, this matrix corresponds to a 2-coloring of the edges of* $K_{7,7}$ *in which there is no* $K_{2,2}$ *of the first color and there is no* $K_{2,3}$ *of the second color. I.e.* $(7,7) \notin R(2,2,(K_{2,2},K_{2,3}))$.

Since M_Y *contains no* 3×2 *submatrices of 0's or 1's, two copies of* M_Y *can be concatenated to produce a* 7×14 *matrix with no* 3×3 *submatrices of 0's or 1's. This implies that* $(7,14) \notin R(2,2,(K_{3,3},K_{3,3}))$. *One can build a* 14×14 *matrix by tensoring a* 2×2 *matrix of 1's with* M_Y. *This matrix shows that* $(14,14) \notin R(2,2,(K_{3,3},K_{3,5}))$.

One can see from the example above that one can formulate a large number of statements of the form $(a,b) \notin R(2,2,A) \implies (a',b') \notin R(2,2,A')$. To do this, one can start with a matrix that doesn't contain certain substructures and use it to build new matrices without certain substructures. This can be a very useful tool when trying to understand a given Ramsey region. Here is one aspect of this idea.

Proposition 1.3.1. *Let* $\mathbb{Z}_n$ *denote the integers modulo* n. *Let* M *be an* $a \times b$ *matrix whose entries are taken from* $\mathbb{Z}_n$. *Suppose* M *contains no* $c \times d$ *submatrix all of whose entries are a given element* $i \in \mathbb{Z}_n$. *Let* $Z_{r,s}$ *denote the* $r \times s$ *matrix of 1's. Then* $Z_{r,s} \otimes M$ *contains no* $((c-1)r+1) \times ((d-1)s+1)$ *submatrix all of whose entries are equal to* i.

Proof. $Z_{r,s} \otimes M$ is the matrix built by stacking r copies of M on top of each other then gluing together s copies of these stacked matrices. The result then follows immediately from the pigeonhole principle. □

Proposition 1.3.2. *The point-block incidence matrix of a* $t-(v,k,\lambda)$ *design contains no* $t \times (\lambda+1)$ *submatrix of 1's and no* $t \times (\bar{\lambda}+1)$ *submatrix of 0's. Where* $\lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$ *and* $\bar{\lambda} = \Sigma_{s=0}^{t}(-1)^s\binom{t}{s}\lambda_s$.

Proof. The statement that the matrix contains no $t \times (\lambda+1)$ submatrix of 1's is clear. The statement that the matrix contains no $t \times (\bar{\lambda}+1)$ submatrix of 0's follows from the fact that the complement to a $t-(v,k,\lambda)$ design is a $t-(v,v-k,\bar{\lambda})$ design [?]. □

Remark 1.3.1. *To produce interesting reduced adjacency matrices, we can start with the incidence matrix of a design. However, proposition 1.3.2 does not guarantee that the incidence matrix of a* $t-(v,k,\lambda)$ *design has no* $(\lambda+1) \times t$ *submatrix of 1's. To establish that a given coloring of the edges of a complete bipartite graph has no monochromatic complete bipartite subgraph, we need to show that the associated reduced adjacency matrix and its transpose contain no monochromatic rectangle of a certain size. Thus, the propositions above are useful but do not complete the picture. However, interesting properties of a matrix do imply interesting properties in the tensor product.*

Designs are very useful in producing points which are not in a given Ramsey region but are typically very close to a boundary point of the region. A standard argument in combinatorics shows that the number of k-sets in a $t-(v,k,\lambda)$ *design is* $b = \lambda\binom{v}{t}/\binom{k}{t}$ *[?]. This fact is sometimes useful to show that a given 2-tuple is in a given Ramsey region by showing that it is*

not possible to have too many blocks (i.e. columns) without forcing certain substructures.

Example 1.3.2. *This example shows that* $(8,8) \in R(2,2,(K_{2,2},K_{3,2}))$. *By the previous paragraphs, this will be established if it is shown that every* 8×8 *0,1-matrix contains either a* 2×2 *submatrix of 1's or a (*2×3 *or* 3×2*) submatrix of 0's. Let* M *be an* 8×8 *0,1-matrix. It is easy to check that if* M *has 3 columns with 4 or more 1's then there will exist a* 2×2 *submatrix of 1's. Therefore, we can assume* M *has 6 or more columns with 5 or more 0's. If there is a column with 6 0's and a column with 5 0's then there will be a* 3×2 *submatrix of 0's. So we can assume* M *has 6 or more columns with exactly 5 0's in each column. The formula for the number of blocks of a* $3-(8,5,1)$ *design yields 5.6 blocks. This is not an integer so such a design does not exist. Furthermore, 5.6 is an upper bound for the number of 5 subsets that can be selected from an 8 element set that avoids the appearance of a 3 element set in more than one block. Since we have 6 or more columns containing 5 0's, there must exist a* 3×2 *submatrix of 0's. Consequently,* $(8,8) \in R(2,2,(K_{2,2},K_{3,2}))$.

1.4 Concluding Remarks

Most of the difficulty of classical Ramsey theory is due to the large step in complexity that occurs between successive Ramsey numbers. With Ramsey regions, the ultimate goal is to compute the fundamental vertex set. This is a large problem but naturally breaks up into a series of much smaller problems. As n increases, fundamental vertices in $F_n(c,(G_1,G_2,\ldots,G_c))$ "converge" to Z_i. The i for which Z_i is a fundamental vertex is the classical Ramsey number $\mathbf{R}(G_1,G_2,\ldots,G_c)$. Information gleaned from one Ramsey

region helps in the understanding of other Ramsey regions. This allows an accumulation of results which can contribute to the solution of a difficult problem such as the determination of classical Ramsey numbers. It is certainly our hope that further connections with other constructions in combinatorics are established and that this allows work in other areas to apply directly to the understanding of Ramsey regions. It is clear that most of this paper can be generalized to hypergraphs in a fairly natural manner. Our goal in a future project is to further develop the algebra involved with Ramsey regions, to describe the algebra of Ramsey regions involving hypergraphs and to extend the algebraic approach initiated in this paper.

Chapter 2

RAMSEY REGIONS FOR HYPERGRAPHS

2.1 Introduction

The work done on Ramsey Regions will now be generalized to the setting of hypergraphs. Many of the definitions and theorems on Ramsey Regions presented in Chapter 1 translate naturally into the setting of hypergraphs. However some theorems, especially those related to constructing Ramsey Regions, require more substantial modifications in order to hold for hypergraphs.

2.2 Main definitions and theorems

Definition 2.2.1. *[?] A* **hypergraph** *is a pair $H = (V, E)$ where V is a set of vertices and E is a set of edges e_i such that $e_i \subseteq V$ for all i. A hypergraph is* **simple** *if there are no repeated edges.*

Notice that our definition does not allow an edge to have a repeated vertex set. This is analogous to not allowing loops in a graph.

Definition 2.2.2. *[?] Let V be a set. The* **powerset** *of V is the set of sets $\mathcal{P}(V) = \{e | e \subseteq V\}$. Let $\mathcal{P}_r(V) = \{e \in \mathcal{P}(V) | \ |V| = r\}$. There is a natural decomposition $\mathcal{P}(V) = \mathcal{P}_0(V) \cup \cdots \cup \mathcal{P}_{|V|}(V)$.*

Every hypergraph in this chapter will be assumed to be simple. Thus, in this chapter, a hypergraph is a pair (V, E) with $E \subseteq \mathcal{P}(V)$. A graph is a hypergraph where $E \subseteq \mathcal{P}_2(V)$.

Definition 2.2.3. *[?] An* **edge n − coloring** *of a hypergraph $H = (V, E)$ is a surjective function $\mathcal{C} : E \to \{c_1, \ldots, c_n\}$.*

Definition 2.2.4. *[?] A hypergraph $H = (V, E)$ is said to be* **k − uniform** *if $E \subseteq \mathcal{P}_k(V)$.*

Definition 2.2.5. *[?] A hypergraph $H = (V, E)$ is said to be* $\mathbf{k-partite}$ *if V can be partitioned into k sets $V_1, \ldots, V_k$ such that for each edge $e \in E$ we have $|e \bigcap V_t| \leq 1$ for all $1 \leq t \leq k$.*

Definition 2.2.6. *[?] A hypergraph $H = (V, E)$ is said to be* **complete** *with respect to properties $P_1, \ldots, P_r$ if the hypergraph is a maximal element in the poset of hypergraphs with vertex set V satisfying all of the properties $P_1, \ldots, P_r$ and partially ordered by the condition $(V, E) < (V, E') \iff E \subset E'$.*

For instance a $k - uniform\ hypergraph\ H = (V, E)$ satisfies $E \subseteq \mathcal{P}_k(V)$ while a *complete* $k - uniform\ hypergraph\ H = (V, E)$ satisfies $E = \mathcal{P}_k(V)$

Definition 2.2.7. *[?] Let V be the vertex set of a hypergraph. $K^r_{n_1,\ldots,n_p}$ will be used to denote a complete r-uniform, p-partite hypergraph which respects a partition $V = V_1 \bigcup \cdots \bigcup V_p$ with $|V_i| = n_i$.*

Example 2.2.1. *The hypergraph $H = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2, 3\}, \{1, 2, 4\}\}$ is a $K^3_{2,1,1}$ with $\{1, 2, 3, 4\} = \{3, 4\} \cup \{2\} \cup \{1\}$.*

Definition 2.2.8. *Let V be the vertex set of a hypergraph with $|V| = n$. If $K_{1,\ldots,1}$ is the complete r-uniform, n-partite hypergraph which respects the partition $V = V_1 \bigcup \cdots \bigcup V_n$ with $|V_i| = 1$ then this will often be denoted K^r_n.*

Example 2.2.2. *The hypergraph $H = (V, E)$ with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ is a $K^3_{1,1,1,1}$ which we can also denote by K^3_4.*

Proposition 2.2.1. *(Basic Facts)*

(1) Every complete r-uniform, p-partite hypergraph has $r \leq p$.

(2) The number of edges in $K^r_{n_1,\ldots,n_p}$ is

$$\sum_{1\leq b_1<b_2<\cdots<b_r\leq p} \prod_{i=1}^{r} n_{b_i}.$$

Definition 2.2.9. *Let $H = (V, E)$ be an r-uniform hypergraph with $|V| = n$. The* **adjacency hypermatrix** *of H is the r-dimensional $n \times n \times \cdots \times n$ hypermatrix, M, with each dimension indexed by the elements of V and with $M_{a_1,a_2,\ldots,a_r} = 1$ if $\{a_1, a_2, \ldots, a_r\} \in E$ and $M_{a_1,a_2,\ldots,a_r} = 0$ if $\{a_1, a_2, \ldots, a_r\} \notin E$.*

Definition 2.2.10. *Let $H = (V, E)$ be an r-uniform hypergraph with $|V| = n$ and let $C : E \to \{c_1, \ldots, c_t\}$ be a t coloring of the edges of E. The colored hypergraph can be represented by an r-dimensional $n \times n \times \cdots \times n$* **colored adjacency hypermatrix**, *M, with each dimension indexed by the elements of V and with entry $M_{a_1,a_2,\ldots,a_r} = C(e_i)$ if $\{a_1, a_2, \ldots, a_k\} = e_i \in E$ and $M_{a_1,a_2,\ldots,a_r} = 0$ if $\{a_1, a_2, \ldots, a_r\} \notin E$.*

Note that if M is an adjacency hypermatrix of a hypergraph or of a colored hypergraph then M is a supersymmetric tensor of format $(n, n, \ldots, n)$. In other words, $M_{a_1,a_2,\ldots,a_r} = M_{a_\pi(1),a_\pi(2),\ldots,a_\pi(r)}$ for every π in the symmetric group S_r.

Let $H = (V, E)$ be a k-uniform, k-partite hypergraph which respects the partition $V = V_1 \cup \cdots \cup V_k$ with $n_i = |V_i|$. Then H can be represented by a k-dimensional *partite* adjacency hypermatrix, $M \in \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_k}$. For each i, the i^{th} dimensional hyper-row is indexed by elements of V_i. We set $M_{a_1,a_2,\ldots,a_r} = 1$ if $\{a_1, a_2, \ldots, a_r\} \in E$ and $M_{a_1,a_2,\ldots,a_r} = 0$ if

$\{a_1, a_2, \ldots, a_r\} \notin E$ and extend this notation to the case of a colored k-uniform, k-partite hypergraph as in the preceding definition. Recall from chapter 1 that in a bipartite graph G, the necessary information to construct G is captured in the reduced adjacency matrix. As a bipartite graph is a 2-uniform 2-partite hypergraph, we are extending this viewpoint to the setting of k-uniform, k-partite hypergraphs. The partite adjacency hypermatrix is sometimes called a reduced adjacency matrix.

Definition 2.2.11. *When $H = (V, E)$ is a k-uniform, k-partite hypergraph, let $RAM_k(H)$ denote the associated $k-$dimensional partite adjacency hypermatrix of H.*

Note that many theorems from chapter 1 immediately carry over to the setting of hypergraphs. These include Proposition 1.2.1, Corollary 1.2.1, Corollary 1.2.2, Corollary 1.2.3, Definition 1.2.1, Proposition 1.2.2, Theorem 1.2.1, Definition 1.2.2, Proposition 1.2.3, Definition 1.2.3, Proposition 1.2.4, and Proposition 1.2.5. We must however, use more care when addressing neighborhood duplication in this more general setting.

Definition 2.2.12. *Let $H = (V, E)$ be a hypergraph and let $v \in V$. The* **neighborhood duplication** *of v in H is the hypergraph $H' = (V', E')$ obtained by introducing a new vertex v' and defining $V' = V \cup v'$ and defining $E' = E \cup S$ where $S = \{e - \{v\} + \{v'\} | e \in E \text{ and } v \in e\}$.*

In other words, the hypergraph H' is formed by distinguishing a vertex v in H, introducing a new vertex v', and requiring that the neighborhood of v' is indistinguishable from the neighborhood of v.

Example 2.2.3. *Let H be the hypergraph $K^3_{1,1,1}$. Any neighborhood duplication of the hypergraph $K^3_{1,1,1}$ yields the hypergraph $K^3_{2,1,1}$. Applying*

neighborhood duplication to $K^3_{2,1,1}$ yields either $K^3_{3,1,1}$ or $K^3_{2,2,1}$ depending one which vertex is chosen to be duplicated.

We will write $G \hookrightarrow H$ if there exists an injective map of G to H. In other words, $G \hookrightarrow H$ if there exists a subgraph of H which is isomorphic to G. We will also write this as $G \subseteq H$.

Theorem 2.2.1. *If H is a hypergraph and H' is obtained from H by neighborhood duplication, then $K^r_n \hookrightarrow H \Leftrightarrow K^r_n \hookrightarrow H'$.*

Proof. It is clear that $K^r_n \hookrightarrow H \implies K^r_n \hookrightarrow H'$. We need to show that $K^r_n \hookrightarrow H' \implies K^r_n \hookrightarrow H$. Let v be the vertex that was duplicated and let v' be the new vertex. If $K^r_n \not\subseteq H$ and $K^r_n \subseteq H'$ then the copy of K^r_n in H' must contain both v and v'. But H and H' do not share an edge whereas all vertices in a complete hypergraph share an edge. □

An immediate useful corollary is the following extension to hypergraphs with a K_n skeleton.

Corollary 2.2.1. *Let $G = (V, E)$ be a hypergraph. Let H and H' be hypergraphs with H' a neighborhood duplication of H. If $|V| = n$ and if $K_n \hookrightarrow G$ then $G \hookrightarrow H \Leftrightarrow G \hookrightarrow H'$.*

Now Corollary 1.2.4 now holds with respect to the new definitions.

Definition 2.2.13. *[?] A proper vertex **k-coloring** of a hypergraph $H = (V, E)$ is a surjective map $C : V \to \{c_1, \ldots, c_n\}$ such that no non-trivial hyperedge has vertices all of the same color. The chromatic number, $\chi(H)$, for a hypergraph H, is the smallest number k such that H admits a proper vertex $k-$coloring.*

Example 2.2.4. *If* $H = K^r_{a_1,\ldots,a_r}$ *then* $\chi(H) = 2$. *To see this, let* $V = V_1 \bigcup \cdots \bigcup V_r$ *be a corresponding vertex partition then map all vertices in* V_1 *to color* c_1 *and map all other vertices to color* c_2. *Every edge of* H *has a vertex in* V_1 *and a vertex not in* V_1. *Thus no edge has vertices all of the same color.*

2.3 Ramsey regions for hypergraphs

We now define Ramsey regions for hypergraphs. Let $(T_1, T_2, \ldots T_c)$ be a c-tuple of sets of k-uniform hypergraphs. Let $R(c, n, (T_1, T_2, \ldots, T_c))$ denote the set of all n-tuples such that every c-coloring of the edges of $K^k_{a_1,a_2,\ldots,a_n}$ forces a monochromatic subgraph of color i from the set T_i (for some i). Ramsey's Theorem guarantees that the n-tuple consisting entirely of 1's is an element of $R(c, n, (T_1, T_2, \ldots, T_c))$ provided that $n \gg 0$. The set of all n-tuples in $R(c, n, (T_1, T_2, \ldots, T_c))$ is called a **Ramsey Region**.

Starting with a (colored) partite adjacency matrix of a lower dimensional hypergraph with special properties, we can sometimes build higher dimensional partite adjacency matrices through stacking, tensor products, and other basic operations. We will see several instances of such constructions in the following pages and, as a result, obtain information on certain specific Ramsey regions.

When dealing with a **complete k − uniform k − partite hypergraph** H with an associated **n − coloring**, each entry in the $RAM_k(H)$ will be c_i (the color) associated to that edge.

Unless otherwise stated assume all graphs are complete, k-uniform, k-partite with an associated n-coloring.

Example 2.3.1. *Let G be a complete n-colored bipartite graph $K^2_{a,b}$ with $a \times b$ partite adjacency matrix M. Assume further that no $r \times r$ submatrix of M is monochromatic. Then the $a \times b \times 2$ hypermatrix M' formed by stacking two copies of M will contain no $r \times r \times 2$ monochromatic submatrix. As a result, M' can be interpreted as the colored partite adjacency matrix of a complete n-colored 3-uniform, 3-partite hypergraph $K^3_{a,b,2}$ which contains no monochromatic $K^3_{r,r,2}$.*

Example 2.3.2. *Let the following matrix correspond to a 2-colored $G = K^2_{4,6}$ (with colors 0 and 1).*

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Note that M contains no 2×2 submatrix consisting of all $1's$ nor one consisting of all $0's$. As a consequence, G contains no monochromatic $K^2_{2,2}$. For any N, if we stack N sheets of matrix M the resulting $4 \times 6 \times N$ hypermatrix, M', will never contain a monochromatic $2 \times 2 \times 2$ submatrix. Interpreting M' as a colored partite adjacency matrix of an edge colored $K^3_{4,6,N}$ we see that there is no monochromatic $K^3_{2,2,2}$. This allows us to conclude that $(4, 6, N) \notin R(2, 3, \{K^3_{2,2,2}\}, \{K^3_{2,2,2}\})$.

We can extend these examples to gain information about Ramsey regions related to *complete $k-uniform$ $k-partite$ hypergraphs* by examining the cases for *complete $(k-1)-uniform$ $(k-1)-partite$ hypergraph.*

Example 2.3.3. *Let H be an n-colored k-uniform k-partite hypergraph which contains no monochromatic $K^k_{r,r,\ldots,r}$. In other words, the colored partite adjacency hypermatrix, M of H, contains no $r \times r \times \cdots \times r$ (k times)*

monochromatic sub-hypermatrix, then by stacking 2 copies of M, we obtain a hypermatrix containing no $r \times r \times \cdots \times r \times 2$ ($k+1$ terms) monochromatic sub hypermatrix thus there exists a n-colored (k+1)-uniform (k+1)-partite hypergraph $\overline{H}$ with no monochromatic $K^{k+1}_{r,r,\ldots,r,2}$.

Note that in the above propositions we looked for square submatrices inside of a parent matrix. Extra care must be taken when applying this procedure to non-square submatrices. For instance, if a matrix M contains a 2×2 submatrix of all $1s$ but no 2×3 submatrix of all $1s$. We cannot expect to "stack" an arbitrary number of this 2×2 submatrix on top of itself and avoid getting a $2 \times 2 \times 3$ submatrix of all $1s$. The reason for this is because we eventually will be forced to place a 2×2 submatrix of all $1s$ on top of another 2×2 submatrix of all $1s$. For the same reason we will eventually be forced to place another 2×2 submatrix of all $1s$ on top of the previous 2. This gives us our $2 \times 2 \times 3$ submatrix of all $1s$.

We can also use the same "stacking" technique for matrices that contain exactly one square submatrix of all $1s$. In this case we cannot stack an arbitrary number of the submatrix, however, we can calculate the number of times the stacking can be carried out before we are forced to place the submatrix on top of itself, thus creating an undesired sub-hypergraph.

Proposition 2.3.1. *Let G be a complete n-colored $K_{a,b}$ bipartite graph which contains exactly one monochromatic $K_{r,r}$ subgraph, then there exists a complete n-colored tri-partite, 3-uniform hypergraph $K^3_{m,n,\binom{m}{r}\binom{n}{r}}$ with no monochromatic $K^3_{r\times r\times 2}$ sub-hypergraph.*

Proof. Let G be as described above. The colored partite adjacency matrix, M of G, contains exactly one $r \times r$ monochromatic submatrix. There are

$\binom{a}{r}$ $\binom{b}{r}$ ways to permute the rows and columns of M such that no two permuted matrices have the $r \times r$ submatrix in the same location within M by stacking these permuted copies of M we obtain an $a \times b \times \binom{a}{r}\binom{b}{r}$ hypermatrix which contains no $r \times r \times 2$ monochromatic sub hypermatrix. Interpreting this as the colored partite adjacency matrix of a $K^3_{a,b,\binom{a}{r}\binom{b}{r}}$ we have obtained a coloring containing no monochromatic $K^3_{r,r,2}$. □

Example 2.3.4. *Consider the following bicolored partite adjacency matrix of a* $K^2_{4,7}$.

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Note that there is exactly one 2×2 *submatrix of all* $1's$ *and exactly one* 2×2 *submatrix of all* $0's$. *We can stack, by permuting rows* $\binom{7}{2}$ *times and columns* $\binom{4}{2}$ *times yielding a* $7 \times 4 \times 168$ *hypermatrix containing no monochromatic* $2 \times 2 \times 2$ *submatrix. Notice that if we then try to stack one more permuted copy of* $RAM_2(G)$ *we will have to form a* $2 \times 2 \times 2$ *submatrix of all* $1s$ *and also one of all* $0s$ *since we have exhausted the possibilities for permuting rows and columns and thus avoiding the undesired submatrix. In this manner we can determine a point on the border of the Ramsey region* $R(2, 3, \{K^3_{2,2,2}\}, \{K^3_{2,2,2}\})$.

Proposition 2.3.2. *Let* $K^k_{a_1,\ldots,a_k}$ *be an n-colored k-uniform k-partite hypergraph with exactly one monochromatic* $K^k_{r\times r\times\cdots\times r}$ *(k times) sub-hypergraph, then there exists an n-colored (k+1)-partite hypergraph* $K^{k+1}_{a_1,\ldots,a_k,\binom{a_1}{r}\cdots\binom{a_k}{r}}$ *with no monochromatic* $K^{k+1}_{r\times r\times\cdots\times r\times 2}$ *(*$k+1$ *terms) sub-hypergraph.*

Proof. Let $G = K^k_{a_1,\ldots,a_k}$ have hyper-edges colored as described above. Then the colored partite adjacency matrix, M of G, contains exactly one monochromatic $r \times r \times \cdots \times r$ submatrix. As a consequence, we can stack $k-$ *dimensional* matrices obtained by permuting each of the hyper-rows in such a way that the $r \times r \times \cdots \times r$ submatrix is displaced with respect to each of the the k dimensions. We can do this a total of $\binom{a_1}{r} \cdots \binom{a_k}{r}$ times without repeating a hypermatrix. In this way we can construct an $a_1 \times a_2 \times \cdots \times a_k \times \binom{a_1}{r} \cdots \binom{a_k}{r}$ hypermatrix from which we may construct the associated n-colored $K^{k+1}_{a_1,\ldots,a_k,\binom{a_1}{r}\cdots\binom{a_k}{r}}$ that contains no monochromatic $K^{k+1}_{r\times r\times\cdots\times r\times 2}$ ($k+1$ terms) sub-hypergraph. □

Chapter 3

PERSISTENCE HOMOLOGY OF SEQUENCES OF NEIGHBORHOOD COMPLEXES FOR GRAPHS

In Chapter three, we will be introducing a construction by which nested families of simplicial complexes can be associated to weighted graphs. The topology of these complexes relate to properties of the graph. Furthermore, the nested families lead to a large source of invariants for the graph. Section 3.1 introduces the basic idea of powerset closure and simplicial complexes. Section 3.2 defines neighborhood complexes and simplicial homology. Section 3.3 introduces the persistence complex as a tool for studying graphs. Section 3.3 is where new results can be found. Sections 3.4 and 3.5 are devoted to in depth examples of computations of persistence complexes.

3.1 Introduction

The following section defines powerset closure and the related topic of simplicial complexes. A more detailed introduction can be found in [?, ?]. Recall from chapter 2 the following definition:

Definition 3.1.1. *[?] Let A be a set. The* **powerset** *of A is the set of sets*

$$\mathcal{P}(A) = \{B | B \subseteq A\}.$$

Let $\mathcal{P}_r(A) = \{B \in \mathcal{P}(A) | \ |B| = r\}$. There is a natural decomposition $\mathcal{P}(A) = \mathcal{P}_0(A) \cup \cdots \cup \mathcal{P}_{|A|}(A)$.

Example 3.1.1. *Let $A = \{a, b, c\}$ then $\mathcal{P}(A) = \mathcal{P}_0(A) \cup \cdots \cup \mathcal{P}_3(A)$ where*

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}, \mathcal{P}_0(A) = \{\varnothing\},$$

$\mathcal{P}_1(A) = \{\{a\}, \{b\}, \{c\}\}, \mathcal{P}_2(A) = \{\{a, b\}, \{a, c\}, \{b, c\}\}$ *and* $\mathcal{P}_3(A) = \{\{a, b, c\}\}$.

Definition 3.1.2. *Let A be a set and let $T \subseteq \mathcal{P}(A)$. The* **powerset closure** *of T is the set*

$$\mathbf{P}(T) = \bigcup_{t \in T} \mathcal{P}(t)$$

Example 3.1.2. *Let* $A = \{a, b, c, d\}$ *and let* $T = \{\{a, b, c\}, \{c, d\}\}$ *then*

$$\mathbf{P}(T) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \varnothing\} \cup \{\{c, d\}, \{c\}, \{d\}, \varnothing\}$$

$$= \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a\}, \{b\}, \{c\}, \{d\}, \varnothing\}.$$

Note that $\mathbf{P}(\mathbf{P}(T)) = \mathbf{P}(T)$.

Definition 3.1.3. $T \subseteq \mathcal{P}(A)$ *is said to be* **powerset closed** *if* $\mathbf{P}(T) = T$

Similar to $\mathcal{P}(A)$, there is a natural decomposition of $\mathbf{P}(T)$ into sets whose entries have constant cardinality, i.e. $\mathbf{P}(T) = \mathbf{P}_0(T) \cup \mathbf{P}_1(T) \cup \ldots$.

Definition 3.1.4. *[?] Let* A *be a set. A set* $T \subseteq \mathcal{P}(A)$ *which is powerset closed is called an (abstract)* **simplicial complex** *in* A*. Elements of* T *are referred to as* **faces** *or* **simplices** *of the simplicial complex.*

Definition 3.1.5. *(Basic Definitions) Let* T *be a simplicial complex.*

(1) If A *is a face of* T *then the* **dimension** *of* A *is defined to be* $dim(A) = |A| - 1$.

(2) The d-**skeleton** *of* T *is defined by* $T^{(d)} = \{x \in T | dim(x) \leq d\}$.

(3) The 1-skeleton of T *is said to be the* **edge skeleton** *of* T.

(4) If v *is a vertex of* T*, the* **star** *of* v *is the set* $star(v) = \{x \in T | v \in x\}$.

Let $A = \{a_1, a_2, \ldots, a_n\}$. Each element of $\mathcal{P}(A)$ is naturally identified with a squarefree monomial in the polynomial ring $K[a_1, a_2, \ldots, a_n]$. For example, the set $\{a, b, c\}$ is naturally identified with the monomial abc. The faces of a simplicial complex, T, form a partially ordered set under set inclusion. The maximal elements of this poset correspond to the maximal

elements in T (geometrically, we think of these as corresponding to the maximal dimensional faces of a realization of the simplicial complex). A generating set for a simplicial complex, T, is a set, U, whose powerset closure is the simplicial complex (i.e. $\mathbf{P}(U) = T$). Every finite simplicial complex has a unique minimal generating set corresponding to the maximal elements in T.

Definition 3.1.6. *(Geometric Realization) Let T be a simplicial complex on n vertices and let the vertices of V be ordered $v_1, v_2, \ldots, v_n$. Let e_i denote the i^{th} coordinate vector of $\mathbb{R}^n$. For each $x \in T$, define $S_x = \{\sum_{v_i \in x} \lambda_i v_i | \lambda_i \in \mathbb{R}^+ \bigcup \{0\}$ and $\sum \lambda_i = 1\}$. Let $S_T = \cup_{x \in T} S_x$. Define a map $G : S_T \to \mathbb{R}^n$ by the linear extension of the map $G(v_i) = e_i$. The image $G(S_T)$ is called a* **standard geometric realization** *of T.*

Definition 3.1.7. *Let T be a simplicial complex on n vertices. Let v_1 be a vertex in T and let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of T. If $g_1(v_1, v_2, \ldots, v_n), g_2(v_1, v_2, \ldots, v_n), \ldots, g_r(v_1, v_2, \ldots, v_n)$ are a set of minimal generators for T then the* **cone of the star of v_1 in T** *is defined to be the simplicial complex, T', on $n+1$ vertices generated by $g_1(v_1^* v_1, v_2, \ldots, v_n), g_2(v_1^* v_1, v_2, \ldots, v_n), \ldots, g_r(v_1^* v_1, v_2, \ldots, v_n)$ where the vertices of T' consist of the vertices of T together with a new vertex v_1^*.*

3.2 Neighborhood Complexes associated to graphs

Definition 3.2.1. *[?] A* **simple graph without loops** *is a pair $G = (V, E)$ where V is a set and $E \subset \mathcal{P}_2(V)$. The set V corresponds to the vertices of G and the set E corresponds to the edges of G. A* **weighted graph** *is a triple $G = (V, E, W)$ where V is a set, $E \subset \mathcal{P}_2(V)$ and W is a*

map $W : E \to \mathbb{R}^+$. *For a given edge,* $e \in E$, $W(E)$ *is called the* **weight** *of* e.

In this chapter, graphs are assumed to be simple without loops. Furthermore, we can think of any graph $G = (V, E)$ as a weighted graph $G = (V, E, W)$ by taking W to be the constant map $W : E \to 1$. We will write this as $G = (V, E, 1)$. The objects of study in this chapter are weighted simple graphs without loops.

Definition 3.2.2. *Let* $v_i, v_j \in V$. *If* $P(v_i, v_j)$ *is a path from* v_i *to* v_j, *define the* **weight** *of* $P(v_i, v_j)$ *to be the sum of the weights of the edges making up the path. The* **distance** *from* v_i *to* v_j, *denoted* $d(v_i, v_j)$, *is defined to be the minimum weight among all paths from* v_i *to* v_j. *If there are no paths between* v_i *and* v_j *then we assign* $d(v_i, v_j) = \infty$.

Definition 3.2.3. *Let* $G = (V, E, W)$ *be a graph and let* $\mathbf{S} \subseteq \mathbf{R} \cup \{\infty\}$. *Given a vertex* $v_i \in V$, *define the* $\mathbf{S}-$**neighborhood** *of the vertex by* $N_S(v_i) = \{v_j \in V \mid d(v_i, v_j) \in \mathbf{S}\}$.

For ease of notation, we typically represent the $\mathbf{S}-$*neighborhood* of a vertex in monomial notation. These monomials are just a shorthand way for representing the associated sets defined above.

Definition 3.2.4. *Given a graph* $G = (V, E, W)$, *a set* $\mathbf{S} \subseteq \mathbf{R} \cup \{\infty\}$ *and a set* $V' \subset V$, *define the* **neighborhood complex,** $\mathbf{N(G, S, V')}$ *to be the power set closure of the union of the* $\mathbf{S}$*-neighborhoods of the vertices in* V'. *In other words* $\mathbf{N(G, S, V')} = \mathbf{P}(\{N_{\mathbf{S}}(v_i) \mid v_i \in V'\})$.

When $V' = V$ we sometimes abbreviate the notation for the neighborhood complex to $\mathbf{N(G, S)}$. For many examples in this chapter, we will be

working with neighborhood complexes for which $0 \notin \mathbf{S}$. When $0 \notin \mathbf{S}$ then we refer to $\mathbf{N(G, S, V')}$ as a **deleted neighborhood complex**.

As a simplicial complex, $\mathbf{N(G, S, V')}$ can be partitioned into sets whose elements have constant cardinality. More precisely, we can write

$$\mathbf{N(G, S, V')} = \mathbf{N_0(G, S, V')} \cup \mathbf{N_1(G, S, V')} \cup \cdots \cup \mathbf{N_t(G, S, V')},$$

where $\mathbf{N_i(G, S, V')} = \{B \in \mathbf{N(G, S, V')} \,|\, |B| = i\}$ and t is the largest cardinality among the entries of $\mathbf{N(G, S, V')}$. For each i, let $\mathbf{V_i} = \mathbf{N_i(G, S, V')} \otimes \mathbf{R}$. In other words, $\mathbf{V_i}$ is defined to be the real vector space with basis indexed by the elements of the set $\mathbf{N_i(G, S, V')}$. Fix an ordering on the elements of V. This will induce a canonical way to express, in monomial notation, each element in $\mathcal{P}(V)$ and therefore, a canonical monomial expression for each element in $\mathbf{N_i(G, S, V')}$. By extension, if $l \in \mathbf{V_i}$ then the terms of l have a canonical monomial expression. For the next definition, we will assume that the elements of V have been ordered.

Definition 3.2.5. *[?] Let $a_0 \ldots a_j$ be a basis element of V_{j+1} (thus, in the ordering on V, $a_0 < a_1 < \cdots < a_j$). Let $\delta : V_{j+1} \to V_j$ be the linear map whose action on basis elements is defined by*

$$\delta(a_0 \ldots a_j) = \sum_{i=0}^{j} (-1)^i (a_0 \ldots \widehat{a_i} \ldots a_j)$$

where $a_0 \ldots \widehat{a_i} \ldots a_q$ is shorthand for the monomial $a_0 \ldots a_i \ldots a_q$ with vertex a_i deleted. We refer to δ as the **Boundary Homomorphism.**

Proposition 3.2.1. *[?] $\delta\delta(a_0 \ldots a_j) = 0$.*

Proof.

$$\delta\delta(a_0 \ldots a_m) = \delta(\sum_{i=0}^{m} (-1)^i (a_0 \ldots \widehat{a_i} \ldots a_m))$$

$$= \sum_{i=0}^{m} (-1)^i \ (\sum_{j=i+1}^{m} (-1)^{j-1} (a_0 \dots \widehat{a_i} \dots \widehat{a_j} \dots a_m))$$

$$+ \sum_{i=0}^{m} (-1)^i \ (\sum_{j=0}^{i-1} (-1)^j (a_0 \dots \widehat{a_j} \dots \widehat{a_i} \dots a_m)) = 0$$

Notice that all terms appear in pairs, once in the first term and once in the second term. That is, for distinct values a and b, at one point $i = a$ and $j = b$, then $i = b$ and $j = a$. The $j - 1$ in the exponent is adjusting for the previously deleted term. [?]

□

Corollary 3.2.1. *If* $l \in \mathbf{V_i}$ *then* $\delta\delta(l) = 0$.

Proof. Follows from the proposition by linearity. □

An immediate consequence of proposition 3.2.1 (and its corollary) is that for each j, $Image(\delta(\mathbf{V_{j+1}})) \subseteq Kernel(\delta(\mathbf{V_j}))$.

Definition 3.2.6. *A* **chain complex** *of vector spaces, $\mathcal{C}$, is a sequence of vector spaces $\dots, V_{-2}, V_{-1}, V_0, V_1, \dots$ connected by homomorphisms $d_n : V_n \to V_{n-1}$, such that $d_n d_{n+1} = 0$ for all n.*

Definition 3.2.7. *A chain complex, $\mathcal{C}$, is said to be* **exact** *if $Ker(\delta(V_i)) = Image(\delta(V_{i+1}))$ for all i.*

Definition 3.2.8. *[?] The* **homology** *at i of a chain complex $\mathcal{C}$ is the quotient space*

$$H_{i-1}(\mathcal{C}) = Ker(\delta(V_i))/Image(\delta(V_{i+1})).$$

Let $h_{i-1}(\mathcal{C})$ denote the dimension of $H_{i-1}(\mathcal{C})$ as a vector space.

Definition 3.2.9. *If $\mathcal{C}, \mathcal{D}$ are chain complexes with boundary maps $\delta_{\mathcal{C}}, \delta_{\mathcal{D}}$ then a* **chain map** *from $\mathcal{C}$ to $\mathcal{D}$ is a collection of linear maps $\pi_i : C_i \to D_i$ such that $\pi_i \circ \delta = \delta \circ \pi_{i+1}$.*

It is important to note that chain maps induce linear maps between the homology spaces of the associated chain complexes.

Let T be a simplicial complex and let $T = T_0 \cup T_1 \cup \cdots \cup T_d$ be its decomposition into sets whose elements have constant cardinality. Let $\mathbf{V_i} = T_i \otimes \mathbb{R}$ and let $d_n = \delta$ (with δ as defined in definition 3.2.5). With a little work, it can be shown that no information is lost if we take $\mathbf{V_0} = 0$ (though strictly speaking our definition implies $\mathbf{V_0} = \mathbb{R}$). From proposition 3.2.1, the $\mathbf{V_i}$ together with the maps δ form a chain complex (with $\delta : \mathbf{V_1} \to \mathbf{V_0}$ appropriately modified). If d is the size of the largest monomial in T, we have

$$0 \xrightarrow{\delta} \mathbf{V_d} \xrightarrow{\delta} \mathbf{V_{d-1}} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathbf{V_1} \xrightarrow{\delta} 0$$

The homology of this complex measures topological attributes of a geometric realization of the simplicial complex, in that $h_q(T)$ can be thought of as measuring the number of $(q+1)$ dimensional holes in a geometric realization of T. For example, $h_0(T)$ tells us the number of connected components and $h_1(K)$ tells us the number of 2-dimensional holes. We will be primarily concerned with the homology neighborhood complexes of graphs.

Lovasz's Theorem is an example of the importance of the interplay between topological and combinatorial information held in a deleted neighborhood complex and information about the original graph.

Theorem 3.2.1. *Lovasz's Theorem (1978) [?]*

Let $G = (V, E, 1)$ be a finite graph. Let $S = \{1\}$. If $\mathbf{N(G, S, V)}$ is $(k-1)$-connected, then G is not $(k+1)$-colorable.

Definition 3.2.10. *[?] Let m be a positive integer and $\mathbf{S}$ an $n-$set, where $n > 2m$. The simple graph whose vertices are the $m-$subsets of $\mathbf{S}$, two being adjacent if they are disjoint, is called the Kneser graph and is denoted by $KG(n, m)$.*

Example 3.2.1. *The following is $KG(5, 2)$ which is isomorphic to the Peterson graph.*

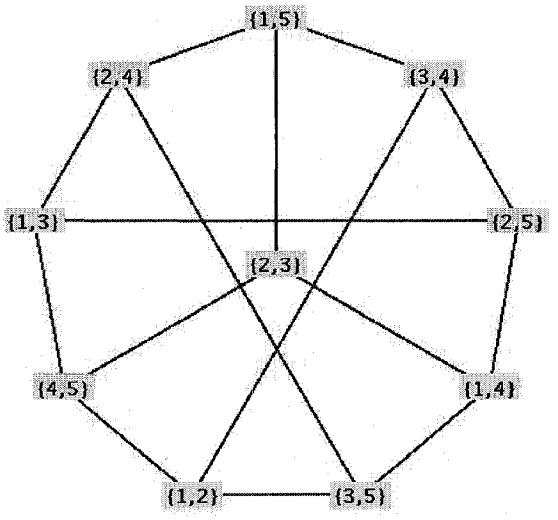

Figure 3.1: $KG(5, 2)$

Example 3.2.2. *The following is an example of a simplicial complex formed from the neighborhood complex of the graph $G = (V, E, 1)$ with $V = \{a, b, c, d\}$, $E = \{ab, bc, cd\}$.*

Figure 3.2: Example of neighborhood complex construction

There are 3 distinct weights that represent shortest paths between pairs of vertices in G. They are $w_1 = 1$, $w_2 = 2$, *and* $w_3 = 3$. *We will now compute the complex for each of* $\mathbf{S}_1 = \{1\}$ *and* $\mathbf{S}_2 = \{1, 2\}$.

$\mathbf{N}(\mathbf{G}, \mathbf{S_1}, \mathbf{V})$ *is generated by* $\{b, ac, bd, c\}$. *As a result, we have* $\mathbf{N}(\mathbf{G}, \mathbf{S_1}, \mathbf{V})$ $= \{ac, bd, a, b, c, d\} = \{ac, bd\} \cup \{a, b, c, d\} \cup \{\varnothing\}$ *and the complex*

$$0 \xrightarrow{\delta_2} \mathbf{V_2} = \langle ac, bd \rangle \xrightarrow{\delta_1} \mathbf{V_1} = \langle a, b, c, d \rangle \xrightarrow{\delta_0} 0$$

where

$$\delta_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \delta_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{S}_2 = \{1, 2\}$. $\mathbf{N}(\mathbf{G}, \mathbf{S_2}, \mathbf{V})$ *is generated by* $\{bc, acd, abd, bc\}$. *As a result, we have* $\mathbf{N}(\mathbf{G}, \mathbf{S_2}, \mathbf{V}) = \{abd, acd, ab, ac, ad, bd, cd, a, b, c, d\} = \{abd, acd\} \cup$ $\{ab, ac, ad, bd, cd\} \cup \{a, b, c, d\}$ *and an associated chain complex*

$$0 \xrightarrow{\delta_3} \mathbf{V_3} = \langle abd, acd \rangle \xrightarrow{\delta_2} \mathbf{V_2} = \langle ab, ac, ad, bd, cd \rangle \xrightarrow{\delta_1} \mathbf{V_1} = \langle a, b, c, d \rangle \xrightarrow{\delta_0} 0$$

where

$$\delta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \delta_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \delta_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}.$$

For each of the other choices for $\mathbf{S}$ (namely $\{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$) we also obtain a neighborhood complex.

3.3 Persistent Homology

Persistent Homology was introduced by Zomorodian, Edelsbrunner, and Letscher in 2002 [?] and further developed in the papers [?, ?, ?, ?, ?], most notably by Zomorodian and Carlsson. It is worth remarking that these works were developed in the setting of data analysis as a tool for analyzing homological aspects of simplicial complexes built from data at varying resolution. The philosophy driving the development was that topological features that persist across a wide range of resolutions were the most pertinent and were less likely to have been introduced through noise or discretization. Weighted (and unweighted) graphs as abstract objects have not previously been approached with these tools.

In my work I take the machinery of persistence complexes, that has been developed in the context of data analysis, and apply the machinery to weighted graphs. These tools have not been used in graph theory before. The persistence complex is a collection of simplicial complexes linked

together through linear maps and it allow both topological and algebraic tools to come into play.

As we shall see, these complexes and their maps are computable and they shed new light on a given graph. Each persistence complex is a graph invariant as is any information derived from the persistence complex. The computational component of persistence complexes has been achieved through a collection of Matlab scripts. The interaction with topology is established through a series of theorems. The overall goal is to connect topological invariants of a persistence complex with properties of the graph from which the complex is built. For instance a graph is bipartite if and only if the zeroth homology of the deleted first neighborhood complex is two dimensional (Theorem 3.3.1). In order to make effective use of these tools, it is important to understand the persistence complexes of some large families of graphs and it is important to know how various modifications of graphs effect the persistence complexes. I also determine what certain values in the associated persistence complex will be for certain weight sets, for example, complete bipartite (Proposition 3.3.5), complete graph (Lemma 3.3.3), complete r-partite graph (Theorem 3.3.4), trees (Corollary 3.3.1), cycles (Proposition 3.3.8).

Understanding the effects that modifications of a graph have on the persistence complex allow one to connect known cases to unknown cases. For instance, removing a sprout corresponds to a certain deformation retract (Proposition 3.3.7) and understanding this fact allows one to understand portions of the persistence complex for trees. A theorem that can be applied in several settings derives from understanding that if a graph is modified by a (sub) neighborhood duplication then the homology of the first deleted

neighborhood complex remain fixed (Theorem 3.3.3). The neighborhood complex of a complement of a graph relates, in a natural way, to the neighborhood complex of the original graph (Proposition 3.3.9) Taking the cone of a graph shifts homology by a dimension for certain neighborhood complexes and relates the homology of a certain complex associated to the cone to the homology of a different complex associated to the complement graph(Theorem 3.3.5).

Thus the goal of this section is to apply and develop a collection of tools that allow one to relate topological invariants of complexes associated to graphs to invariants of the graph and to understand the effect that graph modifications have on the homology of associated complexes. Specifically we will build a structure called a *Persistence Complex* that consists of a family of chain complexes together with a collection of chain maps. The chain maps induce homomorphisms between the homology groups of the corresponding chain complexes. The homology groups of the chain complexes, together with the homomorphisms between homology groups induced by the chain maps, fit together into an algebraic structure called the *Persistence Homology* of the family. This algebraic structure will give us information about the graph as well as provide a highly detailed graph invariant.

The starting point is a weighted graph G. When we speak of a graph without mentioning the weights it will be understood we are assigning a weight of 1 to each edge. Next we apply the Floyd-Warshal algorithm [?] to find the distance between all distinct pairs of vertices in G (recall that the distance between two vertices is the smallest weight path between the vertices).

The output of the Floyd-Warshal algorithm applied to $G = (V, E, W)$ can be stored as the entries of a symmetric matrix (with rows and columns indexed by the elements of V). Suppose we find that there are n distinct values that appear in the matrix. Let w denote the collection of possible weights. Enumerate these values w_1, w_2, ..., w_n (with no a priori assumptions on the ordering of the w_i). Consider the sequence of sets $\mathbf{S}_1 = \{w_1\}$, $\mathbf{S}_2 = \{w_1, w_2\}$, ..., $\mathbf{S}_n = \{w_1, w_2, w_3, \ldots, w_n\}$. Note that $\mathbf{S}_1 \subseteq \mathbf{S}_2 \subseteq S_3 \subseteq \cdots \subseteq \mathbf{S}_n$. In general, each ordering of the elements $w_1, \ldots, w_n$ will yield a different nested sequence of sets. For each $\mathbf{S}_i$ we can construct a neighborhood simplicial complex $\mathbf{N(G, S_i, V)}$ and its associated chain complex. Homology can then be used to measure the failure of exactness for each chain complex of vector spaces. If $\mathbf{S}_i \subseteq S_j$ then $\mathbf{N(G, S_i, V)} \subseteq \mathbf{N(G, S_j, V)}$. More precisely, $\mathbf{N_t(G, S_i, V)} \subset \mathbf{N_t(G, S_j, V)}$ for each t. If we let $\pi_i : \mathbf{N_t(G, S_i, V)} \hookrightarrow \mathbf{N_t(G, S_j, V)}$ then the collection of π_i form a chain map between the chain complexes.

If we form the chain complexes associated to $\mathbf{N(G, S_i, V)}$ for each i and add in the maps π_i, corresponding to inclusion maps, then we may present all of this data in the form of a two dimensional array of vector spaces. The chain maps induce maps between the homology spaces of each chain complex and the corresponding array of spaces and maps is the persistence complex for the sequence $\mathbf{S}_1, \ldots, \mathbf{S}_n$.

More generally, we could consider a much larger collection of complexes. For instance, for any $A \in \mathcal{P}(w)$ and any $B \in \mathcal{P}(V)$, there is a neighborhood complex $\mathbf{N(G, A, B)}$. Furthermore, if $A_1, A_2 \in \mathcal{P}(w)$ and $B_1, B_2 \in \mathcal{P}(V)$ with $A_1 \subseteq A_2, B_1 \subseteq B_2$ then we have $\mathbf{N_t(G, A_1, B_1)} \subset \mathbf{N_t(G, A_2, B_2)}$ and corresponding chain maps. The totality of this data yields a large and detailed graph invariant. For simplicity of exposition and presentation, we will

be restricting ourselves to neighborhood complexes of the form $\mathbf{N(G, S_i, V)}$. In this setting, ascending chains of subsets will yield data that can be presented in a homology table with rows indexed by $\mathbf{S}_i$. Since we only use information about the shortest path between vertices it is not difficult to construct distinct graphs with the same homology table. However, we will see that certain graph characteristics translate into properties of the homology table. That is, we can gain information about the graph by examining the associated homology table.

I wrote matlab code that will carry out much of this procedure from the data $G = (V, E, W)$. The output of the code is the generators of the simplicial complex for each weight and all the associated homologies. This allows data on graphs to be collected quickly in relation to the amount of time needed to perform such calculations by hand.

For a nonweighted graph $\mathbf{G} = (V, E)$ we will associate the weighted graph $G = (V, E, 1)$ (i.e. the weighted graph that results from assigning a weight of 1 to every edge in G). Whenever we discuss paths between vertices it is assumed we are referring to (one of) the shortest distance path(s).

Definition 3.3.1. *Let $G = (V, E, W)$ be a weighted graph. For any pair of vertices $v_i, v_j \in V$, define the distance between v_i and v_j to be the minimum total weight among paths from v_i to v_j. The distance between v_i and v_j will be denoted $d(v_i, v_j)$.*

Definition 3.3.2. *Let $G = (V, E, W)$ be a weighted graph on n vertices and let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices. The* **shortest distance matrix**, *$M(G)$,* **associated** *to G is the matrix whose i, j^{th} entry is $d(v_i, v_j)$.*

It is clear that this matrix will be symmetric.

Definition 3.3.3. *Define the* **distance set** *of a graph* $\mathbf{G}$ *to be*

$$w = \{d(v_i, v_j) \mid v_i, v_j \in V\}.$$

This set is exactly the collection of values that appear in $M(G)$. The deleted distance set, w^* will be the distance set with the zero element removed.

Definition 3.3.4. *Define a* **distance sequence** *of a graph* G *to be a sequence of sets,* $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n$ *where each term in the sequence is a subset of the distance set of the graph and where* $\mathbf{S}_i \subseteq \mathbf{S}_j$ *whenever* $i \leqslant j$.

Definition 3.3.5. *Define the* **natural distance sequence** *of a graph* G *to be the sequence of sets*

$$\mathbf{S}_1 = \{a_1\}, \mathbf{S}_2 = \{a_1, a_2\}, \ldots, \mathbf{S}_n = \{a_1, a_2, \ldots, a_n\}$$

where $\{a_1, a_2, \ldots, a_n\}$ *is the deleted distance set of* G *with* $a_1 < a_2 < \cdots < a_n$.

Unless otherwise stated, if no distance sequence is specified then the natural distance sequence is assumed.

Definition 3.3.6. *Let* $A = A_1, A_2, \ldots, A_n$ *be a distance sequence for a graph* $G = (V, E, W)$. *The* **persistence complex** *associated to* $(\mathbf{G}, \mathbf{A})$ *is the two dimensional array of vector spaces whose* i, j^{th} *entry is* $H_j(\mathbf{N}(\mathbf{G}, \mathbf{A_i}, \mathbf{V}))$. *In addition, the array is equipped with maps* $\pi_{i,j} : H_j(\mathbf{N}(\mathbf{G}, \mathbf{A_i}, \mathbf{V})) \rightarrow H_j(\mathbf{N}(\mathbf{G}, \mathbf{A_{i+1}}, \mathbf{V}))$ *where* $\pi_{i,j}$ *is the map induced by the inclusion* $\mathbf{N}(\mathbf{G}, \mathbf{A_i}, \mathbf{V}) \hookrightarrow \mathbf{N}(\mathbf{G}, \mathbf{A_{i+1}}, \mathbf{V})$.

Definition 3.3.7. *The* **persistence complex table** *associated to* $(\mathbf{G}, \mathbf{A})$ *is the table whose* i, j^{th} *entry is* $h_j(\mathbf{N}(\mathbf{G}, \mathbf{A_i}, \mathbf{V}))$. *We will let* $PCT(i, j)$ *denote the* i, j^{th} *entry of the persistence complex table. In other words,* $PCT(i, j) = H_j(\mathbf{N}(\mathbf{G}, \mathbf{A_i}, \mathbf{V}))$.

It is clear that for a given G and A, both the persistence complex (and the corresponding persistence complex table) are graph invariants. For a graph G with n vertices the PCT derived from the natural distance sequence will have $(n-1)$ columns.

Proposition 3.3.1. *Let* A *be the deleted distance set for a connected graph,* $G = (V, E, W)$. *If* G *has* n *vertices then* $PCT(i, n-1) = 1$ *if* $i \in \{1, n-1\}$ *and* $PCT(i, n-1) = 0$ *if* $i \notin \{1, n-1\}$.

Proof. The neighborhood complex $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ consists of the n distinct monomials of length $(n-1)$. This complex is the triangulation of a hypersphere of dimension $n-2$. The proposition is simply a statement about the homology spaces of such a complex. □

Proposition 3.3.2. *For a given* $(\mathbf{G}, \mathbf{A})$ *and associated PCT with* k *columns, if there exists* i *such that* $PCT(i, k) = 1$ *then* $PCT(j, k) = 1$ *for* $j \geqslant i$. *Furthermore,* $PCT(m, k) \geqslant PCT(n, k)$ *for* $n \geqslant m$.

Proof. For a given $\mathbf{S}_i$, h_0 tells us the number of connected components of $N(G, \mathbf{S}_i)$. If a is in the generator monomial for vertex x in $\mathbf{N}(\mathbf{G}, \mathbf{S_i})$, then a will be in the generator monomial in $\mathbf{N}(\mathbf{G}, \mathbf{S_j})$ for $j \geqslant i$. This tells us the monomials of a complex can only gain vertices. Thus the associated simplicial complex can only become more connected. Thus the number of connected components can only decrease. □

Recall that a graph G is bipartite if its vertices can be partitioned into two sets V_1 and V_2 such that there are no edges spanning a pair of vertices in V_1 and there are no edges spanning a pair of vertices in V_2. In other words, every edge joins a vertex in V_1 with a vertex in V_2. A complete bipartite graph, $K_{m,n}$ is a bipartite graph with $|V_1| = m$ and $|V_2| = n$ with a maximal number of edges [?].

If G can be partitioned into k sets, $V_1,\ldots,V_k$, such that no two vertices in the same V_i are adjacent then we say that G is k-partite.

Theorem 3.3.1. *Let $G = (V, E, 1)$ be a connected graph and let $A = \{1\}$. For the pair (G, A),*

(1) $h_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 2$ iff G is bipartite

(2) $h_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 1$ iff G is not bipartite.

Proof. First, note that $h_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V}))$ is the number of connected components in the edge skeleton of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$. Second, note that a graph is bipartite if and only if it has no odd cycles. Suppose $v_1, v_2 \in V$ and that there is an even length path from v_1 to v_2. Consider the sequence $v_1, a_0, a_1, a_2, \ldots, a_{2n}, v_2$ enumerating, in order, the vertices encountered in the even length path. Then $\{v_1, a_1\}, \{a_1, a_3\}, \{a_3, a_5\}, \ldots, \{a_{2n-1}, v_2\}$ are edges are in the simplicial complex $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$. As a result, v_1 and v_2 are in the same connected component of the edge skeleton of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$. More precisely, two vertices v_1, v_2 are in the same connected component of the edge skeleton of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ if and only if there is a path of even length between v_1 and v_2.

Suppose that G is connected and bipartite and that V_1, V_2 is the corresponding disjoint decomposition of V. If a path starts at a vertex in V_1

and has an even number of steps then it will end at a vertex in V_1. By connectedness of G, there is a path between every pair of vertices in V_1. By bipartiteness, any path between two vertices in V_1 is of even length. As a consequence, all the vertices in V_1 are in the same connected component of the edge skeleton of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$. Similarly for the vertices in V_2. Furthermore, no vertex in V_1 is connected by a path of even length to a vertex in V_2. Therefore, the edge skeleton of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ has two connected components and $h^0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 2$.

Suppose that G is connected and is not bipartite. Then there exists an odd cycle in G. Let $v_1, v_2 \in V$. Let P_1 denote a path from v_1 to a vertex w_1 contained in an odd cycle. Let P_2 denote a path from w_1 to v_2. Let P_3 denote an odd length cycle from w_1 to w_1. Then either the path $P_1 + P_2$ or the path $P_1 + P_3 + P_2$ is of even length since both are paths from v_1 to v_2 and the difference in lengths of the two paths is the length of P_3 which is odd. □

Proposition 3.3.3. *Let $G = (V, E, W)$ and $G' = (V, E', W')$ be two graphs on the same vertex set V. Suppose $M(G) = M(G')$. If A is any distance sequence then the persistence complex for (G, A) and the persistence complex for (G', A) are identical.*

Proof. Monomials in neighborhood complexes are determined by the shortest distance matrix. □

Proposition 3.3.4. *(Homology of Spheres and Balls) [?]*

(1) Let S^n be a simplicial complex corresponding to a triangulation of an n-dimensional sphere. Then $H^i(S^n) = 1$ if $i = 0, n$ and $H^i(S^n) = 0$ if $i \neq 0, n$.

(2) *Let B^n be a simplicial complex corresponding to a triangulation of an n-dimensional ball. Then $H^i(B^n) = 1$ if $i = 0$ and $H^i(B^n) = 0$ if $i \neq 0$*

Proposition 3.3.5. *Let $G = (V, E, 1)$ be a complete bipartite graph, $K_{a,b}$, on n vertices. Let $A_1 = \{1\}$, let $A_2 = \{1, 2\}$ and let $A_3 = \{2\}$. Then the homology of the associated neighborhood complexes is:*

$$H_i(\mathbf{N}(\mathbf{G}, \mathbf{A_j}, \mathbf{V})) = \begin{cases} H_i(B^{a-1} \cup B^{b-1}) & \text{for j} = 1 \\ H_i(S^{n-2}) & \text{for j} = 2 \\ H_i(S^{a-2} \cup S^{b-2}) & \text{for j} = 3 \end{cases}$$

Proof. Since G is a bipartite graph, we can decompose $V = V_1 \cup V_2$ with $|V| = n, |V_1| = a, |V_2| = b$. Since G is a *complete* bipartite graph, each vertex in V_1 is connected to every vertex in V_2 (and vice versa). Since every edge in G has weight 1, $\mathbf{N}(\mathbf{G}, \mathbf{A_1}, \mathbf{V})$ consists of 2 generators, a monomial containing every vertex in V_1 and a monomial containing every vertex in V_2. The simplicial closure of each of these generators corresponds to a simplicial complex of a triangulation of a $a-1$ dimensional ball and a disjoint simplicial complex of a triangulation of a $b-1$ dimensional ball. Since every vertex is either distance 1 or 2 from every other vertex and is distance 0 from itself, $\mathbf{N}(\mathbf{G}, \mathbf{A_2}, \mathbf{V})$ is generated by all the square free monomials of degree $n-1$ in the vertices of V. The corresponding simplicial complex is the triangulation of a $n-2$ dimensional sphere. In a similar manner, $\mathbf{N}(\mathbf{G}, \mathbf{A_3}, \mathbf{V})$ is generated by all the square free monomials of degree $a-1$ in the variables of V_1 and by all the square free monomials of degree $b-1$ in the variables of V_2. As a consequence, the corresponding simplicial complex is the triangulation of a disjoint union of an $a-2$ dimensional sphere and a $b-2$ dimensional sphere. □

Definition 3.3.8. *Let A, X be topological spaces with $A \subseteq X$. Let $d :\rightarrow X \times [0,1] \rightarrow X$ be a continuous map. Suppose for every $x \in X$ and $a \in A$ the following conditions are satisfied: $d(x,0) = x, d(x,1) \in A, d(a,t) = a$. Then d is called a* **deformation retraction** *and A is said to be a* **deformation retract** *of X. If X deformation retracts to a point then X is said to be* **contractible**.

If A is a deformation retract of X then we think of X as a space that can be retracted down to A. The deformation retraction is a map which captures this idea of continuously shrinking X down to A.

Lemma 3.3.1. *Let $G(T)$ be a geometric realization of a simplicial complex, T, on n vertices. Let v_1 be a vertex in T and let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of T. Then $G(T)$ is a deformation retract of the geometric realization, $G(T')$, of the simplicial complex T' that is the cone of the star of v_1 in T.*

Proof. Consider the map $d : G(T') \times [0,1] \rightarrow G(T')$ given by $d(\lambda_i' v_1' + \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, t) = (1-t)\lambda_i' v_1' + (t\lambda_i' + \lambda_1)v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$. Then d satisfies the conditions of a deformation retract from T' to T. □

Rather than work with the geometric realization of a simplicial complex, we will always work directly with the simplicial complex. For instance, in the setting above, we will say that T is a deformation retract of T'. The following theorem allows a simplification of many arguments.

Theorem 3.3.2. *[?] (Mayer-Vietoris Exact Sequence for Simplicial Complexes) Let K, L be simplicial complexes. There is a long exact sequence of*

homology groups

$$\begin{array}{ccccc} \to & H_n(K\cap L) & \to & H_n(K)\oplus H_n(L) & \to & H_n(K\cup L) \\ \to & H_{n-1}(K\cap L) & \to & H_{n-1}(K)\oplus H_{n-1}(L) & \to & H_{n-1}(K\cup L) \\ & \vdots & & \vdots & & \vdots \\ \to & H_0(K\cap L) & \to & H_0(K)\oplus H_0(L) & \to & H_0(K\cup L)\to 0 \end{array}$$

As a first application of the Mayer-Vietoris exact sequence we show:

Lemma 3.3.2. *Let T be a simplicial complex with generators $g_1, g_2, \ldots g_n$. Let $S \subseteq T$ be a subcomplex with generators $h_1, h_2, \ldots, h_m$. Suppose that S is contractible. Let v be a new vertex and let T' be the simplicial complex generated by $g_1, g_2, \ldots, g_n, vh_1, vh_2, \ldots, vh_m$. Then $H_i(T) = H_i(T')$ for all i.*

Proof. If K is the simplicial complex generated by $vh_1, vh_2, \ldots, vh_m$ then K is a cone over the simplicial complex $h_1, h_2, \ldots, h_m$ and thus is contractible. Let L be the simplicial complex generated by $g_1, g_2, \ldots, g_n$ (thus $L = T$). Note that $K \cap L = S$ and thus is contractible. Finally $K \cup L = T'$. Applying the Mayer-Vietoris exact sequence we get:

$$\begin{array}{ccccc} \to & 0 & \to & 0\oplus H_n(T) & \to & H_n(T') \\ \to & 0 & \to & 0\oplus H_{n-1}(T) & \to & H_{n-1}(T') \\ & \vdots & & \vdots & & \vdots \\ \to & 0 & \to & 0\oplus H_1(T) & \to & H_1(T') \\ \to & H_0(S) & \to & H_0(K)\oplus H_0(T) & \to & H_0(T')\to 0 \end{array}$$

□

We immediately obtain that $H_i(T) = H_i(T')$ for $i \geq 2$. Note that S and K are both connected thus $H_0(S) = H_0(K) = \mathbb{R}$. Further note that since T and T' have the same number of connected components then $H_0(T) = H_0(T')$. As a consequence, we obtain that $H_1(T) = H_1(T')$.

Proposition 3.3.6. *(Homology of Deformation Retracts) [?] Let T, S be simplicial complexes with T a deformation retract of S. Then $H^i(S) = H^i(T)$ for all i.*

Definition 3.3.9. *Let $G = (V, E, 1)$ be a graph. Let $v \in V$. A* **(sub) neighborhood duplication** *of v is a graph $G' = (V', E', 1)$ where $V' = V \cup v^*$ and $E' \subseteq E \cup \{\{v^*, v_i\} | \{v, v_i\} \in E\}$.*

Theorem 3.3.3. *Let $G = (V, E, 1)$ be a graph. Let $A = \{1\}$. Let G' be a (sub) neighborhood duplication of a vertex $v \in V$. Then $H_i(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = H_i(\mathbf{N}(\mathbf{G'}, \mathbf{A}, \mathbf{V'}))$ for all i.*

Proof. The neighborhood complex $\mathbf{N}(\mathbf{G'}, \mathbf{A}, \mathbf{V'})$ can be written as the union of a simplicial complex K that is the union of a cone of a subcomplex of the star of v in $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ and the simplicial complex $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$. The intersection of these two simplicial complexes is a subcomplex of the star of v in $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ that itself is a cone. Applying the lemma above (or directly applying the Mayer-Vietoris sequence) yields the result. □

Lemma 3.3.3. *Let $G = (V, E, 1)$ be a complete graph, K_n, on n vertices. Let $A = \{1\}$. Then $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ is a simplicial complex corresponding to the triangulation of an $n - 2$ dimensional sphere.*

Theorem 3.3.4. *Let $G = (V, E, 1)$ be a complete r-partite graph, $K_{a_1, a_2, \ldots, a_r}$, on n vertices (with $r > 2$). Let $A_1 = \{1\}, A_2 = \{1, 2\}, A_3 = \{2\}$. Then the homology of the associated neighborhood complexes is:*

$$H_i(\mathbf{N}(\mathbf{G}, \mathbf{A_j}, \mathbf{V})) = \begin{cases} H_i(S^{r-2}) & \text{for j} = 1 \\ H_i(S^{n-2}) & \text{for j} = 2 \\ H_i(S^{a_1-2} \cup S^{a_2-2} \cup \cdots \cup S^{a_r-2}) & \text{for j} = 3 \end{cases}$$

Proof. When $j = 1$ we not that $K_{a_1,a_2,\ldots,a_r}$ is obtained from $K_{1,1,\ldots,1}$ by a series of neighborhood duplications. The result for $j = 1$ follows from the preceding lemmas. When $j = 2$, $\mathbf{N(G, A_j, V)}$ is generated by all square free monomials of degree $n-1$ on n variables thus $\mathbf{N(G, A_j, V)}$ is the simplicial complex associated to a triangulation of an $n-2$ dimensional hypersphere. Let $V_1, V_2, \ldots, V_r$ be a partitioning of the vertex set V such that each vertex in V_i is connected to every vertex in V_j by an edge in G (with $i \neq j$) and where no vertex in V_i is connected by an edge to any other vertex in V_i. When $j = 3$, $\mathbf{N(G, A_j, V)}$ is generated by the disjoint union, over i, of all square free monomials in V_i of degree $a_i - 1$. Thus $\mathbf{N(G, A_j, V)}$ is the simplicial complex associated to a triangulation of the disjoint union of hyperspheres $S^{a_1-2} \cup S^{a_2-2} \cup \cdots \cup S^{a_r-2}$. □

Definition 3.3.10. *Let $G = (V, E, 1)$ be a graph. Let $G' = (V', E', 1)$ be a graph built from G by adding a new vertex, v, and a new edge $\{v, v_i\}$ for some $v_i \in V$. Then G' is called a sprout of G.*

Proposition 3.3.7. *Let $G = (V, E, 1)$ be a connected graph on 2 or more vertices. Let G' be a sprout of G. Let $A = \{1\}$, then $\mathbf{N(G', A, V')} = \mathbf{N(G', A, V)}$ and $\mathbf{N(G, A, V)}$ is a deformation retract of $\mathbf{N(G', A, V)}$.*

Proof. Suppose G has n vertices. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of V. If $g_1, g_2, \ldots, g_n$ are the A-neighborhoods of the vertices in G then they generate $\mathbf{N(G, A, V')}$. If the sprouted vertex is v and if v is connected to v_1 then $v_1, vg_1, g_2, \ldots, g_n$ generate $\mathbf{N(G', A, V')}$. Since G is a connected graph, the generator v_1 is redundant, thus $\mathbf{N(G', A, V')} = \mathbf{N(G', A, V)}$. Since v appears only in one generator of $\mathbf{N(G', A, V)}$ and since the other generators are identical, the simplicial complex generated by

$vg_1, g_2, \ldots, g_n$ differs from the simplicial complex generated by $g_1, g_2, \ldots, g_n$ by taking a cone over a face. Since a face is contractible to a point, we are done. □

Corollary 3.3.1. *Let $G = (V, E, 1)$ be a connected tree that contains at least 2 vertices. Let $A = \{1\}$. Then $H_i(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 2$ if $i = 0$ and $H_i(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 0$ if $i \neq 0$.*

Proof. A connected tree on 2 or more vertices can be built from a connected tree on 2 vertices through a series of sprouts. Thus the neighborhood complex for a connected tree deformation retracts to a neighborhood complex for a connected tree on two vertices. The neighborhood complex for a connected tree on two vertices is two points. □

Proposition 3.3.8. *Let $G = (V, E, 1)$ be a cycle of length n with $n > 4$. Let $A = \{1\}$. Then $H_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = H_1(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 2$ if n is even and $H_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = H_1(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 1$ if n is odd.*

Proof. Suppose n is even. Then G is bipartite. Label the vertices in G such that the edges are of the form $\{v_i, v_{i+1}\}_{i<n} \cup \{v_n, v_1\}$. In $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$, the generator corresponding to a vertex will consist of the neighbors on each side of the vertex. In other words, $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ is generated by the monomials $\{v_n v_2, v_1 v_3, v_2 v_4, v_3 v_5, v_4 v_6, \ldots, v_{n-2} v_n, v_{n-1} v_1\}$. The corresponding simplicial complex consists of two cycles, one on the even indexed vertices and one on the odd indexed vertices. Thus $H_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = H_1(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 2$.

Suppose i is odd. Then G is not bipartite and $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ consists of a single cycle. Thus $H_0(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = H_1(\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})) = 1$. □

Definition 3.3.11. *Let $G = (V, E, 1)$ be a graph. The complement graph to G is $G^\vee = (V, E^\vee, 1)$ where $E^\vee = \mathcal{P}_2(V) - E$.*

Proposition 3.3.9. *Let $G = (V, E, 1)$ be a graph and let $G^\vee$ be its complement. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertex set V. Let $A = \{1\}$. Let g_i be the generator of $\mathbf{N}(\mathbf{G}, \mathbf{A}, \mathbf{V})$ coming from vertex v_i and let $g_i^\vee$ be the generator of $\mathbf{N}(\mathbf{G}^\vee, \mathbf{A}, \mathbf{V})$ coming from the same vertex. Let $h = \prod_j v_j$. Then $g_i^\vee g_i = h/v_i$.*

Proof. Let $v_i \in V$. For $j \neq i$, vertex v_j is distance one away from v_i in either G or $G^\vee$ (but not in both). On the other hand, v_i is distance zero from v_i in both G and $G^\vee$. The proposition is just a formulation of this statement. □

For a graph $G = (V, E, 1)$ we can form a new graph $Cone(G) = (V', E', 1)$ by introducing a new vertex , v^*, for which $V' = V \cup v^*$ and $E' = E \cup \{v_i, v^*\}_{v_i \in V}$. In other words, $Cone(G)$ is the original graph G with one additional vertex added sharing an edge with every vertex in G.

Theorem 3.3.5. *Let $G = (V, E, 1)$ be a connected graph on n vertices. Let $A_1 = \{1\}, A_2 = \{1, 2\}, A_3 = \{2\}$. Then the homology of the associated neighborhood complexes of $G' = Cone(G)$ is:*

$$H_i(\mathbf{N}(\mathbf{G}', \mathbf{A_j}, \mathbf{V})) = \begin{cases} H_i(B^0) & \text{for j} = 1 \\ H_i(B^0) & \text{for j} = 2 \\ H_i(\mathbf{N}(\mathbf{G}^\vee, \mathbf{A_1}, \mathbf{V})) & \text{for j} = 3 \end{cases}$$

and

$$H_i(\mathbf{N}(\mathbf{G}', \mathbf{A_j}, \mathbf{V}')) = \begin{cases} H_{i-1}(\mathbf{N}(\mathbf{G}', \mathbf{A_j}, \mathbf{V})) & \text{for j} = 1, \text{i} > 1 \\ H_0(\mathbf{N}(\mathbf{G}, \mathbf{A_j}, \mathbf{V})) - 1 & \text{for j} = 1, \text{i} = 1 \\ 1 & \text{for j} = 1, \text{i} = 0 \\ H_i(S^{n-1}) & \text{for j} = 2 \\ H_i(\mathbf{N}(\mathbf{G}^\vee, \mathbf{A_1}, \mathbf{V})) & \text{for j} = 3 \end{cases}$$

Proof. First note that the only deleted distance sets for G' are given by A_1, A_2, A_3 since the diameter of G' is the minimum of 2 and the diameter of G. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices in V. Let $g_1, g_2, \ldots, g_n$ be the generators of $\mathbf{N(G, A_1, V)}$ given by the n vertices in V. Then $\mathbf{N(G', A_1, V)}$ is generated by $v^*g_1, v^*g_2, \ldots, v^*g_n$. In other words, the simplicial complex $\mathbf{N(G', A_1, V)}$ is a cone over the simplicial complex $\mathbf{N(G, A_1, V)}$ and is therefore contractible. As a result, $H_i(\mathbf{N(G', A_1, V)})$ has the homology of a point.

The neighborhood complex $\mathbf{N(G', A_2, V)}$ is generated by all square free monomials of degree n in the variables $\{v_1, \ldots, v_n, v^*\}$ with the exception of the monomial $\prod_i v_i$. As a consequence, $\mathbf{N(G', A_2, V)}$ is topologically equivalent to a triangulated hypersphere with one face removed. By the Mayer-Vietoris sequence, such a space is contractible and has the homology of a point.

Let g_1 be the generator of $\mathbf{N(G', A_3, V)}$ associated to vertex $v_1 \in V$. Then g_1 is the monomial of all vertices exactly distance 2 away from v_1 in G'. Every vertex is either distance 1 or 2 away from v_1. The vertices which are distance 1 away from v_1 in G' are the cone vertex v together with the vertices that were distance 1 away from v_1 in G. All other vertices in G' are distance 2 away from v_1. These are precisely the same vertices which are distance 1 away from v_1 in $G^\vee$. Thus, the generators $g_1, g_2, \ldots, g_n$ of the neighborhood complex $\mathbf{N(G', A_3, V)}$ are the same as the generators of the neighborhood complex $\mathbf{N(G^\vee, A_1, V)}$.

In order to understand the homology of the neighborhood complexes of $\mathbf{N(G', A_j, V')}$ we need to understand the effect of the extra generator of the complex associated to the cone vertex, v. For A_3, the generator

associated to v is $h = 0$. For A_1 and A_2, the generator associated to v is $h = \prod_j v_j$. As a consequence, $H_i(\mathbf{N}(\mathbf{G}', \mathbf{A_3}, \mathbf{V}')) = H_i(\mathbf{N}(\mathbf{G}^\vee, \mathbf{A_1}, \mathbf{V}))$ and $H_i(\mathbf{N}(\mathbf{G}', \mathbf{A_2}, \mathbf{V}')) = H_i(S^{n-1})$. For A_1, the generator associated to v is $h = \prod_j v_j$. We can decompose $\mathbf{N}(\mathbf{G}', \mathbf{A_1}, \mathbf{V}')$ into the union of the two simplicial complexes $K = \mathbf{N}(\mathbf{G}', \mathbf{A_1}, \mathbf{V})$ and $L = \prod_j v_j$. Since both K and L are contractible, the result follows from the Mayer-Vietoris exact sequence of homology. □

3.4 Introductory Example of a Persistence Complex

The following is an example of a persistence complex constructed from the graph $G = (V, E, 1)$ with vertex set $V = \{a, b, c\}$ and $E = \{ab, bc\}$.

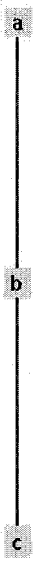

Figure 3.3: Example of Persistence Complex

The only two distances between pairs of vertices in G are 0,1,2. Let $A_1 = \{1\}$ and let $A_2 = \{1, 2\}$. Consider the distance sequence $\{A_1, A_2\}$. The generators for $\mathbf{N(G, A_1, V)}$ are b, ac, and b. That is because with distance 1, a can reach b, b can reach a and c, and c can reach b. Decomposing $\mathbf{N(G, A_1, V)}$ into sets of monomials where each set consists of all of the monomials of a fixed degree, we get

$$\mathbf{N(G, A_1, V)} = \{ac\} \cup \{a, b, c\} \cup \{\varnothing\}.$$

In a similar manner,

$$\mathbf{N(G, A_2, V)} = \{ab, ac, bc\} \cup \{a, b, c\} \cup \{\varnothing\}$$

since every vertex is distance one or two from every other vertex in G.

The associated persistence complex of vector spaces is the diagram:

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\mathbf{A}} & \mathbb{R}_1 = \langle ac \rangle & \xrightarrow{\mathbf{B}} & \mathbb{R}_3 = \langle a, b, c \rangle & \xrightarrow{\mathbf{C}} & 0 \\ & & \downarrow \mathbf{G} & & \downarrow \mathbf{H} & & \\ 0 & \xrightarrow{\mathbf{D}} & \mathbb{R}_3 = \langle ab, ac, bc \rangle & \xrightarrow{\mathbf{E}} & \mathbb{R}_3 = \langle a, b, c \rangle & \xrightarrow{\mathbf{F}} & 0 \end{array}$$

where

$$\mathbf{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{H} = I_3$$

It should be noted that $\mathbf{HB} = \mathbf{EG}$.

Notice that our persistence complex is really two chain complexes "stacked" on top of each other. These two complexes are connected by the inclusion maps $\mathbf{G}$ and $\mathbf{H}$.

The maps within the chain complexes are constructed by applying the boundary homomorphism to the elements of the vector space. For example, map $\mathbf{E}$ takes the monomials of length 2, assigns a column to each one, and applies the boundary homomorphism to that monomial. This construction is illustrated in the table below. Note that the map $\mathbf{E}$ appears in the lower right corner of the table, the columns are indexed by the monomials of length 2 and the rows are indexed by the monomials of length 1.

	ab	ac	bc
a	-1	-1	0
b	1	0	-1
c	0	1	1

Now we can compute the homology at various points in the chain complexes.

Examine the simplicies that arise from w_1. These are ac, a, b, and c. Constructing this object K_1, with vertex set $\{a, b, c\}$ and edge set $\{ac\}$, gives a topological object with 2 components and no 2-dimensional holes. This implies that $h_0(K_1) = 2$ and $h_1(K_1) = 0$. We confirm these values computationally using the ranks of the maps involved.

$$h_0(K_1) = \#columns(\mathbf{C}) - rank(\mathbf{C}) - rank(\mathbf{B}) = 3 - 0 - 1 = 2$$

$$h_1(K_1) = \#columns(\mathbf{B}) - rank(\mathbf{B}) - rank(\mathbf{A}) = 1 - 1 - 0 = 0$$

Similarly if we construct K_2 from the simplicies that arise from w_2 we get a triangle. That is, vertex set $\{a, b, c\}$ and edge set $\{ab, ac, bc\}$. We see that this object has 1 component and 1 2-dimensional hole (the inside of the triangle). Again, we confirm the homology implications computationally.

$$h_0(K_2) = \#columns(\mathbf{F}) - rank(\mathbf{F}) - rank(\mathbf{E}) = 3 - 0 - 2 = 1$$

$$h_1(K_2) = \#columns(\mathbf{E}) - rank(\mathbf{E}) - rank(\mathbf{D}) = 3 - 2 - 0 = 1$$

We will also keep track of a basis for each homology space found above and note the persistence of the elements of the homology space.

$$Ker(\mathbf{C})/Image(\mathbf{B}) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle / \left\langle \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \simeq \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

$$Ker(\mathbf{F})/Image(\mathbf{E}) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle / \left\langle \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\rangle \simeq \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

The inclusion maps G and H induce maps between the homology groups of the complexes. In particular, the inclusion map H induces the map

$$Ker(\mathbf{C})/Image(\mathbf{B}) \xrightarrow{\mathbf{M}} Ker(\mathbf{F})/Image(\mathbf{E})$$

where in the basis chosen for each homology space,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

As we move down map $\mathbf{H}$ we get a surjection of spaces.

We can also examine the map induced by $\mathbf{G}$.

$$Ker(\mathbf{B})/Image(\mathbf{A}) = \langle \begin{bmatrix} 0 \end{bmatrix} \rangle / \langle \begin{bmatrix} 0 \end{bmatrix} \rangle = \langle \begin{bmatrix} 0 \end{bmatrix} \rangle$$

$$Ker(\mathbf{E})/Image(\mathbf{D}) = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle / \langle \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rangle = \langle \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rangle$$

In this case, we have an injection.

Thus the associated PCT to our example is

$$PCT = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$$

and with rows and columns labeled

$$\left(\begin{array}{c|cc} & h_1 & h_0 \\ \hline A_1 & 0 & 2 \\ A_2 & 1 & 1 \end{array}\right)$$

3.5 Second Example of a Persistence Complex

Let $G = C_{14}$ under the natural distance sequence.

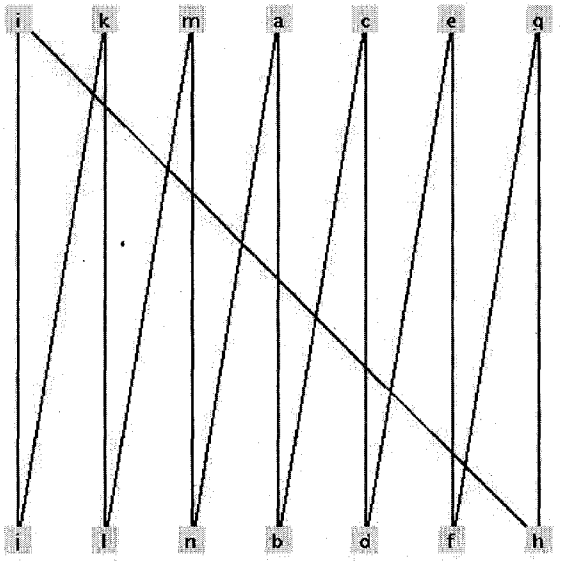

Figure 3.4: Second example of persistence complex

Then $PCT(G) =$

	h_{12}	h_{11}	h_{10}	h_9	h_8	h_7	h_6	h_5	h_4	h_3	h_2	h_1	h_0
S_1	0	0	0	0	0	0	0	0	0	0	0	2	2
S_2	0	0	0	0	0	0	0	0	0	0	1	2	1
S_3	0	0	0	0	0	0	0	0	0	1	1	1	1
S_4	0	0	0	0	0	0	0	0	1	2	0	0	1
S_5	0	0	0	0	0	0	0	3	2	0	0	0	1
S_6	0	0	0	0	0	0	0	1	0	0	0	0	1
S_7	1	0	0	0	0	0	0	0	0	0	0	0	1

We will focus on the values from the $PCT(G)$ that appear in the column associated to h_1. That is $PCT(1, 12) = 2$, $PCT(2, 12) = 2$, $PCT(3, 12) = 1$, and $PCT(i, 12) = 0$ for $i > 3$.

The part of the persistence complex associated to the values of interest to us is

$$\begin{array}{ccccccccc}
S_1 = \{1\} & \ldots & \longrightarrow & 0 & \xrightarrow{\mathbf{A}} & \mathbb{R}_{14} & \xrightarrow{\mathbf{B}} & \mathbb{R}_{14} & \longrightarrow & 0 \\
 & & & & & \downarrow \mathbf{G} & & & & \\
S_2 = \{1,2\} & \ldots & \longrightarrow & \mathbb{R}_{56} & \xrightarrow{\mathbf{C}} & \mathbb{R}_{56} & \xrightarrow{\mathbf{D}} & \mathbb{R}_{14} & \longrightarrow & 0 \\
 & & & & & \downarrow \mathbf{H} & & & & \\
S_3 = \{1,2,3\} & \ldots & \longrightarrow & \mathbb{R}_{196} & \xrightarrow{\mathbf{E}} & \mathbb{R}_{84} & \xrightarrow{\mathbf{F}} & \mathbb{R}_{14} & \longrightarrow & 0
\end{array}$$

where

$$B = \left(\begin{array}{c|cccccccccccccc}
 & ac & am & bd & bn & ce & df & eg & fh & gi & hj & ik & jl & km & ln \\ \hline
a & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
f & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
g & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
h & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
m & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
n & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Let K_i be the topological object formed by monomials generated for S_i

$$h_1(K_1) = \#columns(\mathbf{B}) - rank(\mathbf{B}) - rank(\mathbf{A}) = 14 - 12 - 0 = 2$$

$$NullSpace(B) = \left(\begin{array}{c|cc}
ac & 1 & 0 \\
am & -1 & 0 \\
bd & 0 & 1 \\
bn & 0 & -1 \\
ce & 1 & 0 \\
df & 0 & 1 \\
eg & 1 & 0 \\
fh & 0 & 1 \\
gi & 1 & 0 \\
hj & 0 & 1 \\
ik & 1 & 0 \\
jl & 0 & 1 \\
km & 1 & 0 \\
ln & 0 & 1
\end{array}\right)$$

Notice that each column corresponds to a cycle that appears in K_1. This combination of monomials will also map to zero in the subsequent

vector spaces for monomials of length 1 but higher values of w_i. Since matrices C and E have a different number of rows than B, these cycles must be translated to become columns that correspond to C and E respectively. For example, we find which rows of C correspond to ac, am, ce, eg, gi, ik, and km. Then we add a new column to C which has an appropriately signed 1 in the rows corresponding to these monomials. Repeat the step for the second cycle and call this new matrix C'. Now by comparing the ranks of C and C' it can be determined if both of these vectors form a basis for the $ker(D)/image(C)$. Then repeat this step for matrix E and call it E', to see if these vectors are in $ker(F)/image(E)$. In other words we are determining the persistence of the vectors which form a basis for the homologies.

In this case in it is seen that $rank(C) = 41$ and $rank(C') = 42$. This means that one of the additional vectors lands in the $image(C)$ while one vector remains in the $ker(D)/image(C)$. Likewise, $rank(E) = 70$ and $rank(E') = 71$. Recall that we are examining the homologies related to the h_1 column of $PCT(G)$. This information implies that there is one vector that persists in all three non-zero homologies for this column. The other vector is killed as we move along G, and another vector appears in $ker(D)/image(C)$ since $PCT(2, 12) = 2$. This new vector is then killed moving along H since $PCT(3, 12) = 1$ and we have seen the other vector survives the entire path.

To verify these results consider

$$\delta(-abd + abe + acn - ade + alm - aln + bce - bcn - cdf + cdg - cfg + deg$$

$$-efh + efi - ehi + fgi - ghj + ghk - gjk + hik - ijl + ijm - ilm + jkm) =$$

$$[ac - am + ce + eg + gi + ik + km] + [-bd + bn - df - fh - hj - jl - ln]$$

$$= (1) * monomials\ in\ column\ one\ of\ NullSpace(B)$$

$$+(-1) * monomials\ in\ column\ two\ of\ NullSpace(B)$$

It is easy to verify that all monomials of length three in the above equation are in K_2.

3.6 A Sampling of Graphs and their PCTs

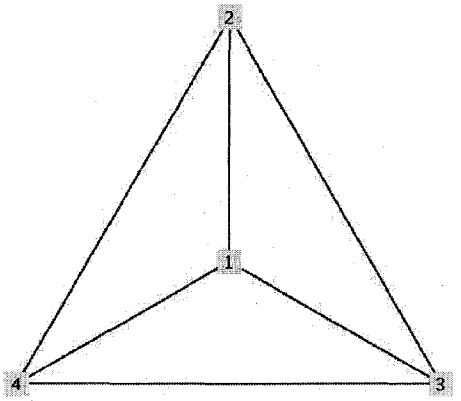

Figure 3.5: K_4

$$PCT(K_4) = \left(\begin{array}{c|ccc} & h_2 & h_1 & h_0 \\ \hline S_1 & 1 & 0 & 1 \end{array}\right)$$

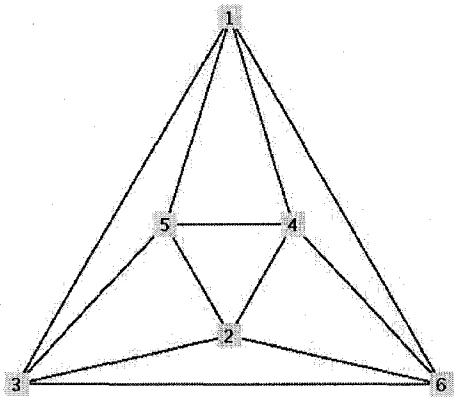

Figure 3.6: Octahedron

$$PCT(Octohedron) = \left(\begin{array}{c|ccccc} & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 1 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

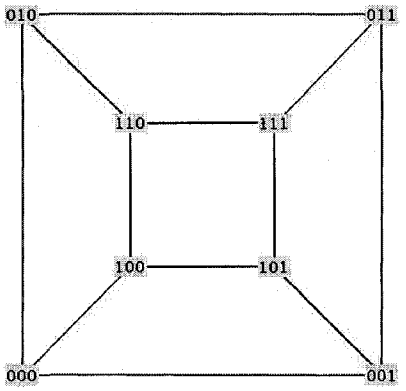

Figure 3.7: Hypercube

$$PCT(HyperCube(3)) = \left(\begin{array}{c|ccccccc} & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ S_2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ S_3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

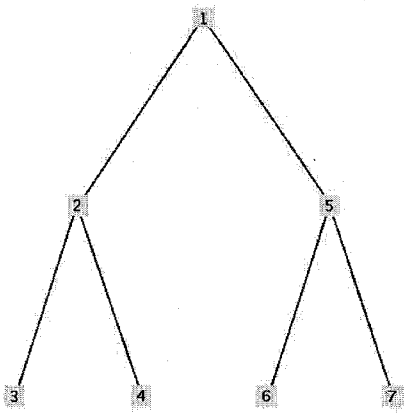

Figure 3.8: Binary Tree

$$PCT(CompleteBinaryTree(2)) = \left(\begin{array}{c|cccccc} & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 2 \\ S_2 & 0 & 0 & 0 & 3 & 0 & 1 \\ S_3 & 0 & 0 & 3 & 0 & 0 & 1 \\ S_4 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

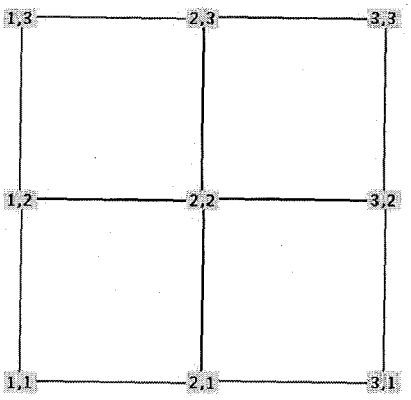

Figure 3.9: Grid Graph (3,3)

$$PCT(gridgraph(3,3)) = \left(\begin{array}{c|cccccccc} & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ S_2 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 1 \\ S_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ S_4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

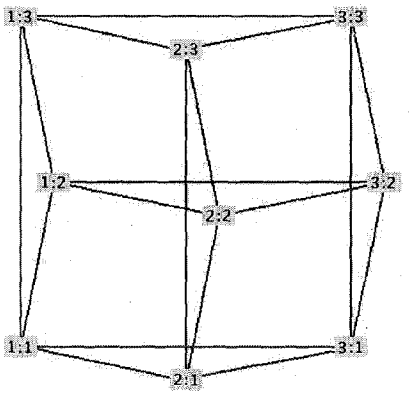

Figure 3.10: Torus Grid Graph (3,3)

$$PCT(TorusGridGraph(3,3)) = \left(\begin{array}{c|cccccccc} & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

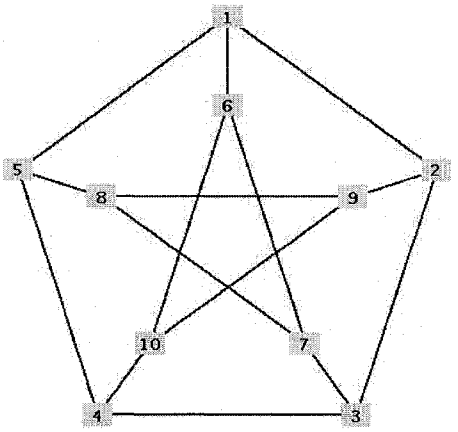

Figure 3.11: Petersen

$$PCT(PetersenGraph) = \left(\begin{array}{c|ccccccccc} & h_8 & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

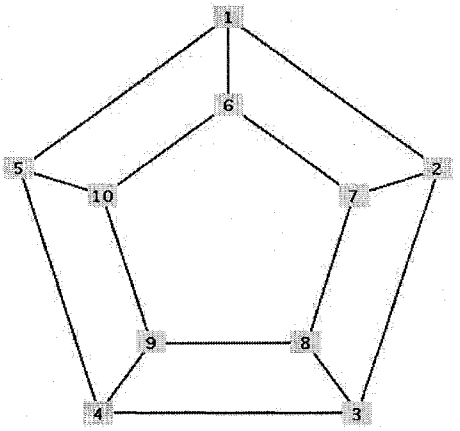

Figure 3.12: Prism (5)

$$PCT(PrismGraph(5)) = \left(\begin{array}{c|ccccccccc} & h_8 & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ S_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

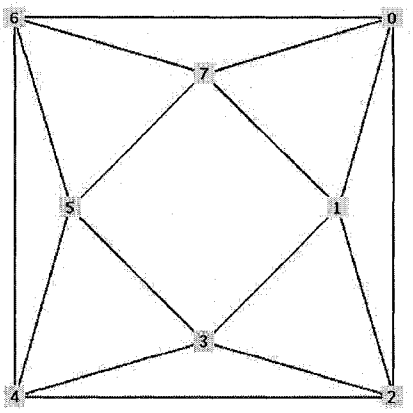

Figure 3.13: AntiPrism (4)

$$PCT(AntiPrismGraph(4)) = \left(\begin{array}{c|ccccccc} & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 4 & 1 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

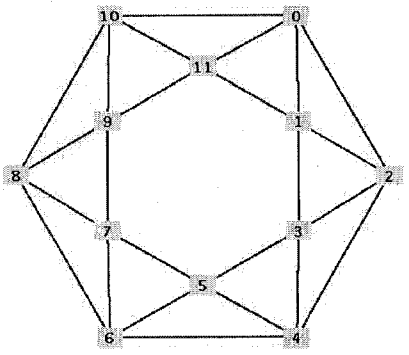

Figure 3.14: AntiPrism (6)

$$PCT(AntiPrismGraph(6)) =$$

$$\left(\begin{array}{c|cccccccccccc} h_{10} & h_{11} & h_{10} & h_9 & h_8 & h_7 & h_6 & h_5 & h_4 & h_3 & h_2 & h_1 & h_0 \\ \hline S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ S_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Appendix A

MATLAB CODE

A.1 Introduction

The following is Matlab code which I wrote to find homologies and construct the persistence complex tables for various graphs. This allowed a large number of examples to be collected, from which patterns in the data could be recognized and researched. This code was necessary since it would be undesirable to do the calculations by hand for even small graphs (with 6+ vertices).

The matrices involved in these calculations can become quite large. Both memory and computing time can become an issue for large graphs (I was never successful running this code on a graph with 20 vertices).

A.2 Driver Script

```
%This is the driver program
clear
clc
tic
A=[Inf     1   Inf   Inf   Inf   Inf   Inf   Inf 1   1
     1   Inf   1     Inf     1   Inf     1   Inf 1   1
   Inf     1   Inf     1   Inf     1   Inf     1 1   1
   Inf   Inf     1   Inf   Inf   Inf   Inf   Inf 1   1
   Inf     1   Inf   Inf   Inf   Inf   Inf   Inf 1   1
   Inf   Inf     1   Inf   Inf   Inf   Inf   Inf 1   1
   Inf     1   Inf   Inf   Inf   Inf   Inf   Inf 1   1
   Inf   Inf     1   Inf   Inf   Inf   Inf   Inf 1   1
   1     1     1     1     1     1     1     1   Inf 1
   1     1     1     1     1     1     1     1   1   Inf];
n=size(A,1); %B=zeros(n,n);
%B(i,j) is the weight of the cheapest path from i to j in A
    B = floydwarshal(A);
    %Floydwarshal finds the shortest path between all pairs of points
    H=zeros(1,1);k=1;
    changecounter = 1; changeposition = zeros(1,2); %error check
    for i=1:n-1
        for j=i+1:n
            %for m=1:k
                H(k)=B(i,j);
                k=k+1;
                if 1==mod(k,500)
                   H=unique(H);
                end
            %end
       end
    end
    H=unique(H);
    %H is now a list of distinct numbers representing shortest
    %paths between pairs of vertices in A
    C=zeros(n,n,size(H,2));
    for i=1:n
        for j=1:n
            for m=1:size(H,2)
                if B(i,j) <= H(m) & i~=j
                    C(i,j,m)=1;
                    %C(i,j,m) = 1 if j is in the
                    %neighborhood generated by i wrt the mth weight
```

```
            end
        end
    end
end
for i=1:size(H,2)
    disp(H(i))
    disp(C(:,:,i))
end
disp('weight of shortest path from i to j')
disp(B)
S=zeros((n+1)*(nchoosek(n,floor(n/2))),n,size(H,2)); rowsofZ=1;
for m=1:size(H,2)
    Z=zeros(1,n);
    for i=1:n
        Z=[Z; powerset(C(i,:,m))];  %concatanating our submonomials
    end
    Z=unique(Z,'rows');
    %Z contains the generated monomials and their submonomials
    temp=zeros(1,n); k=1;
    for i=1:size(Z,1)
        if sum(Z(size(Z,1)-i+1,:)) > 0
            temp(k,:) = Z(size(Z,1)-i+1,:);k=k+1;
        end
    end
    Z=temp;
    if size(Z,1) > rowsofZ
        rowsofZ=size(Z,1);
    end
    for s=1:size(Z,1)
        for t=1:n
            S(s,t,m) = Z(s,t);
        end
    end
end
temp=zeros(rowsofZ,size(S,2),size(S,3));
for i=1:size(temp,1)
    for j=1:size(temp,2)
        for s=1:size(temp,3)
            temp(i,j,s) = S(i,j,s);
        end
    end
end
S=temp;
Cn=zeros(nchoosek(n,floor(n/2)),n,n-1,size(C,3)); k=ones(1,n-1);
%Cn(i,:,k,m) is a monomial of length k found wrt weight m.
%this is a seperation of the monomials into sets of their respective weights.
for t=1:size(C,3)
    k=ones(1,n-1);
     for i=1:n-1
        for s=1:size(S,1)
            if sum(S(s,:,t)) == i
                 Cn(k(i),:,i,t) = S(s,:,t); k(i)=k(i)+1;
             end
         end
     end
end

    j = n;
    At = [];
    packets = 1000;
        for i = 1:j-1
            N = nchoosek(j,i);
                while size(At,1) < N
                    newvec = round(rand(N,j));
```

```
                    temp   = [];
                        for ci = 1:N
                            if sum( newvec(ci,:) ) == i
                                temp = [temp; newvec(ci,:)];
                            end
                        end
                        At = unique([At; temp],'rows');
                    end
                    %disp(At)
                    Atemp=zeros(size(At,1),size(At,2));
                    for pan=1:size(At,1)
                        Atemp(pan,:)=At(size(At,1)+1-pan,:);
                    end
                    At = Atemp;
                    %disp(At)
                    eval(['At',num2str(i),' = At;']);
                    %disp(['i = ',num2str(i),':
                    %N = ',num2str(N),' and size(A) =
                    %[',num2str(size(A)),'].'])
                    At = [];
                end

    for i=1:size(Cn,4)
        for j=1:size(Cn,3)
            i;
            j;
            Simplex=border(Cn(:,:,j,i));
            temp1=trimCn(Cn,j,i);
            s=size(temp1,1);
            t=size(temp1,2);
            temp2=zeros(size(Simplex,1),s);
            for a=1:size(temp2,1)
                for b=1:size(temp2,2)
                   if a <= size(Simplex,1) & b<= size(Simplex,2)
                    temp2(a,b)=Simplex(a,b);
                   end
                end
            end
            temp2;
        Simplex2=temp2; %this is the variable to check the [0],[] column number error
            Simp(j,i).mat=Simplex2;
            Simp(j,i).mat;
%             if j == size(Cn,3)
%                 if Simp(j,i).mat == [0]
%                     Simp(j,i).mat = []
%                 end
%             end

            if j > 1
                if Simp(j,i).mat == [0]
                    Simp(j,i).mat = [];
                    changeposition(changecounter,1) = j;
                    changeposition(changecounter,2) = i;
                    changecounter = changecounter + 1;
                end
            end
            %Begining of test part
            array=zeros(1,n);
            placekeeper=0;
           %change starts
           if j>1
               for p=1:j-1
                   array(p)=1;
               end
```

```
                                        %array just for debugging
                %comb=perms(array);
                %comb=unique(comb,'rows');
                %comb1=zeros(size(comb,1),size(comb,2));
                %for p=1:size(comb,1)
                %    comb1(p,:)=comb(size(comb,1)+1-p,:);
                %end
                %comb=comb1;
                placekeeper=zeros(1,size(eval(['At',num2str(j-1),]),1));
                counter2=1;
            end
            %change ends
                if j>1
                    for p=1:size(eval(['At',num2str(j-1),]),1)
                        counter1=0;
                        for q=1:size(Cn,1)
                            if eval(['At',num2str(j-1),'(p,:)'])==Cn(q,:,j-1,i)
                                counter1=counter1+1;
                            end
                        end
                        if counter1==0
                            placekeeper(1,counter2)=p;counter2=counter2+1;
                        end
                    end
                end
                placekeeper;
                counter1=0;
                for p=1:size(placekeeper,2)
                    if placekeeper(1,p)~=0
                        counter1=counter1+1;
                    end
                end
                counter2=0;
                for p=1:size(placekeeper,2)
                    if placekeeper(1,p) > 0
      eval('Simp(j,i).mat(placekeeper(p)-counter2,:)=[]','Simp(j,i).mat=[];');
                        counter2=counter2+1;
                    end
                end
                %end of test part
Simplex2;
Simp(j,i).mat;
         end
     end
for i=1:size(Simp,2)
    Simp(1,i).mat=zeros(1,size(Simp(1,i).mat,2));
end
Homology=zeros(size(Simp,1),size(Simp,2));
%Homology(i,j) gives the homology of weight i between elts of length j+1 to
%length j
for i=1:size(Simp,2)
    for j=1:size(Simp,1)-1
        Homology(j,i)=size(Simp(j,i).mat,2)-rank(Simp(j+1,i).mat)-rank(Simp(j,i).mat);
    end
end
Homology;
for i=1:size(Homology,2)
Homology(size(Homology,1),i)
=size(Simp(size(Simp,1),i).mat,2)-rank(Simp(size(Simp,1),i).mat);
end
Homology;
clc
disp(A)
for i=1:size(H,2)
```

```
        weight = (H(i))
        disp(C(:,:,i))
        degreeofconnectedness = connectedfct7(Cn(:,:,:,i))
end
Homology
tyme = toc
%save('ATESTWORKSPACE')
```

A.3 Floyd-Warshall Algorithm

```
function [short] = floydwarshal(A)
n = size(A,1);
 Dk=zeros(n,n,n);  %D(i,j,k) is the i,j position in Dk
 for i=1:n
     for j=1:n
         Dk(i,j,1)= min(A(i,j), A(i,1) + A(1,j));
     end
 end
 for k=2:n
     for i=1:n
         for j=1:n
             Dk(i,j,k) = min(Dk(i,j,k-1), Dk(i,k,k-1) + Dk(k,j,k-1));
         end
     end
 end
 short=Dk(:,:,n)
```

A.4 Create Powerset

```
function B = powerset(A)
%Will take a vector representing a set and find the powerset
n=size(A,2);
s=sum(A); values=zeros(1,1);
k=1;
for i=1:n
    if A(i)~=0
        values(k)=i;
        k=k+1;
    end
end
k=0; B=zeros(1,n);
for i=1:size(values,2)-1
    T=nchoosek(values,i);
    for j=1:size(T,1)
        k=k+1;
        for m=1:size(T,2)
            B(k,T(j,m))=1;
        end
    end
end
k=k+1;
B(k,:)=A;
```

A.5 Apply Boundary Map

```
function [Simplex] = border(C)

T=sum(C,2);k=1;
    for i=1:size(C,2)
        if T(i) > k
            k=2;
        end
    end
if k > 1
m=size(C,1);n=size(C,2);k=zeros(nchoosek(n,floor(n/2)),m);
v=zeros(1,n);
for i=1:n
    v(i)=i;
end
K=nchoosek(v,sum(C(1,:),2)-1);
A=zeros(m,sum(C(1,:),2));
Simplex=zeros(size(K,1),m);
for i=1:m
    placeholder=1;
    for j=1:n
        if C(i,j)~=0
            A(i,placeholder)=j;placeholder=placeholder+1;
        end
    end
end
A;
for t=1:size(A,1)
    Asubp=zeros(sum(C(1,:),2),sum(C(1,:),2));
        for i=1:size(Asubp,1)
        Asubp(i,:)=A(t,:);
        end
        for i=1:size(Asubp,1)
        Asubp(i,i)=0;
        end
Asubp;
Asubpp=zeros(size(Asubp,1),size(Asubp,2)-1);
for i=1:size(Asubp,1)
    r=1;
    for j=1:size(Asubp,2)
        if Asubp(i,j)~=0
            Asubpp(i,r)=Asubp(i,j);r=r+1;
        end
    end
end
Asubpp;
for i=1:size(Asubpp,1)
    for j=1:size(K,1)
        if K(j,:)==Asubpp(i,:)
            for q=1:size(Asubp,2)
                if Asubp(i,q)==0
                    qp=q;
                end
            end
            if 0==mod(qp,2)
                Simplex(j,t)=-1;
            elseif 1==mod(qp,2)
                Simplex(j,t)=1;
            end
        end
    end
end
K;
```

```
end
Simplex;  %Simplex has columns ordered in reverse lexocographic ordering
%Simplex had rows listed in lexicographic ordering
size(K,1);
elseif k==1
Simplex=zeros(5,5);
end
```

A.6 Store Generating Monomials

```
function [A] = trimCn(Cn,r,p)
%takes Cn(:,:,i,j) as an input and trims it to correct size matrix
if sum(sum(Cn(:,:,r,p),1),2) > 0
temp1 = sum(Cn(:,:,r,p),1); h2=sum(Cn(:,:,1,p),1);
h=zeros(1,size(Cn,2)); A=Cn(:,:,r,p);
    for i=1:size(temp1,2)
        if temp1(i)==0 & h2(i)==0
        %this means we want to eliminate that vertex from the matrix
            Atemp=zeros(size(A,1),size(A,2)-1);
            for s=1:size(A,1)
                for t=1:size(A,2)
                    if t < i
                        Atemp(s,t)=A(s,t);
                    elseif t > i
                        Atemp(s,t)=A(s,t-1);
                    end
                end
            end
            A=Atemp;
        end
    end
    h3=sum(A,2);k=2; Aprime=A(1,:);
    for i=2:size(A,1)
        if h3(i) > 0
            Aprime=[Aprime; A(i,:)];
        end
    end
     A=Aprime;

else A=[0];
end
A;
```

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